MUS420 Supplement
Stability Proof for a Cylindrical Bore with Conical Cap

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Outline

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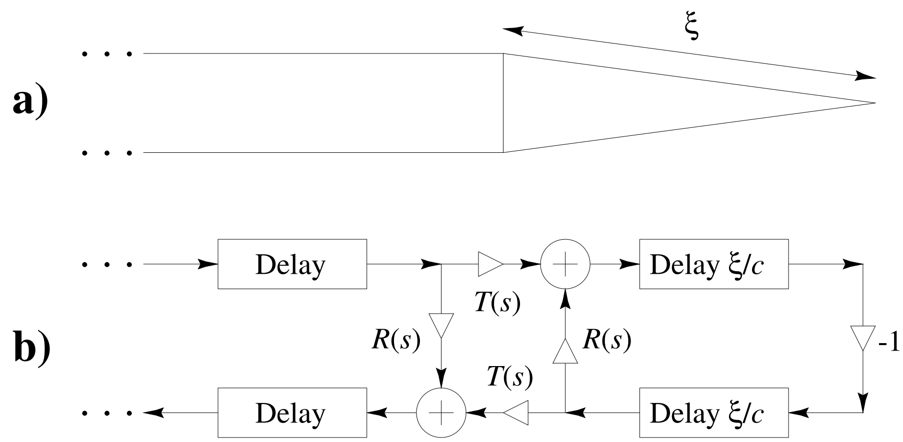
Cylinder with Conical Cap



- Cylinder open or closed on left side
- Otherwise closed
- Obviously passive physically
- Hard to show! [$R(s)$ and $T(s)$ are unstable]

Scattering Filters at the Cylinder-Cone Junction

Wave impedance at frequency ω rad/sec in a converging cone:

$$Z_\xi(j\omega) = \frac{\rho c}{S(\xi)} \cdot \frac{j\omega}{j\omega - c/\xi} \qquad \text{(converging cone impedance)}$$

where

$$\begin{aligned}\xi &= \text{distance to the apex of the cone}\\ S(\xi) &= \text{cross-sectional area of cone}\\ \rho c &= \text{wave impedance in open air}\end{aligned}$$

In the limit as $\xi \to \infty$,

$$Z_\infty(j\omega) = \frac{\rho c}{S} \qquad \text{(cylindrical tube impedance)}$$

Reflectance of the conical cap, *seen from cylinder*:

$$R(s) = -\frac{c/\xi}{c/\xi - 2s}$$

Transmittance to the right:

$$T(s) = 1 + R(s) = -\frac{2s}{c/\xi - 2s}$$

- $R(s)$ and $T(s)$ are first-order transfer functions, each having a single real pole at $s = c/(2\xi) \Rightarrow$ *unstable*
- $R(s)$ and $T(s)$ identical from left and right given no wavefront area discontinuity.

Reflectance of the Conical Cap

- Let $t_\xi \stackrel{\Delta}{=} \xi/c$ denote the time to propagate across the length of the cone in one direction
- Reflectance of complete (lossless) cone is -1 for pressure waves
  (reflects like an open-ended cylinder)
- Round-trip transfer function from cone entrance to tip and back is

$$R_{t_\xi}(s) \stackrel{\Delta}{=} -e^{-2st_\xi} = e^{-st_\xi}(-1)e^{-st_\xi}$$

(reflectance seen *inside* the cone)

Reflectance of Conical Cap Seen from Cylinder

From the figure, we can derive the conical cap reflectance to be

$$
\begin{aligned}
R_J(s) &= \frac{R(s) + 2R(s)R_{t_\xi}(s) + R_{t_\xi}(s)}{1 - R(s)R_{t_\xi}(s)} \\
&= \frac{1 + (1 + 2st_\xi)R_{t_\xi}(s)}{2st_\xi - 1 - R_{t_\xi}(s)} \\
&= \frac{1 - (1 + 2st_\xi)e^{-2st_\xi}}{2st_\xi - 1 + e^{-2st_\xi}} \\
&\stackrel{\Delta}{=} \frac{N(s)}{D(s)}
\end{aligned}
$$

For very large t_ξ, the conical cap reflectance approaches $R_J = -e^{-2st_\xi}$ which coincides with the impedance of a length $\xi = ct_\xi$ open-end cylinder, as expected.

Stability Proof Outline

- A transfer function $R_J(s) = N(s)/D(s)$ is stable if there are no poles in the right-half s plane. That is, for each zero s_i of $D(s)$, we must have re$\{s_i\} \leq 0$. If this can be shown, along with $|R_J(j\omega)| \leq 1$, then the reflectance R_J is shown to be passive.
- We must also study the system zeros (roots of $N(s)$) in order to determine if there are any pole-zero cancellations (common factors in $D(s)$ and $N(s)$).

- Since re $\{st_\xi\} \geq 0$ if and only if re $\{s\} \geq 0$, for $t_\xi > 0$, we may set $t_\xi = 1$ without loss of generality. Thus, we need only study the roots of

$$\begin{aligned} N(s) &= 1 - e^{-2s} - 2se^{-2s} \\ D(s) &= 2s - 1 + e^{-2s} \end{aligned}$$

If this system is stable, we have stability also for all $t_\xi > 0$.

- Since e^{-2s} is not a rational function of s, the reflectance $R_J(s)$ may have infinitely many poles and zeros.

Stability Proof

First consider the roots of the denominator

$$D(s) = 2s - 1 + e^{-2s}.$$

At any pole (solution s of $D(s) = 0$), we must have

$$s = \frac{1 - e^{-2s}}{2}$$

To obtain separate equations for the real and imaginary parts, take the real and imaginary parts of $D(\sigma + j\omega) = 0$ to get

$$\begin{aligned} \text{re}\{D(s)\} &= (2\sigma - 1) + e^{-2\sigma}\cos(2\omega) = 0 \\ \text{im}\{D(s)\} &= 2\omega - e^{-2\sigma}\sin(2\omega) = 0 \end{aligned}$$

Both of these equations must hold at any pole of the reflectance. For stability, we further require $\sigma \leq 0$. Defining $\tau = 2\sigma$ and $\nu = 2\omega$, we obtain the simpler conditions

$$\begin{aligned} e^{\tau}(1-\tau) &= \cos(\nu) \\ e^{\tau} &= \frac{\sin(\nu)}{\nu} \end{aligned}$$

For any poles of $R_J(s)$ on the $j\omega$ axis, we have $\tau = 0$, and the second equation reduces to sinc$(\nu) = 1$. It is well known that the sinc function is less than 1 in magnitude at all ν except $\nu = 0$. Therefore, this relation can hold only at $\omega = \nu = 0$, and so

Any right-half-plane poles occur at $\omega = 0$.

Stability Proof, continued

The same argument can be extended to the entire right-half plane as follows. Going back to

$$\frac{\sin(\nu)}{\nu} = e^{\tau},$$

since $|\sin(\nu)/\nu| \leq 1$ for all real ν, and since $|e^{\tau}| > 1$ for $\tau > 0$, this equation clearly has no solutions in the right-half plane. Therefore,

Any right-half-plane poles occur at $s = 0$.

A Pole at DC

Since both of the conditions

$$\begin{aligned} e^{\tau}(1-\tau) &= \cos(\nu) \\ e^{\tau} &= \frac{\sin(\nu)}{\nu} \end{aligned}$$

are clearly satisfied for $\tau = \nu = 0$, we see that there is in fact a pole in the reflectance at dc ($s = 0$), provided it is not canceled by a zero at dc in the numerator $N(s)$.

The Left-Half Plane

In the left-half plane, there are many potential poles:

- The first of the two equations
  $$e^{\tau}(1-\tau) = \cos(\nu)$$
  has infinitely many solutions for each $\tau < 0$, since the elementary inequality $1 - \tau \leq e^{-\tau}$ implies
  $$e^{\tau}(1-\tau) < e^{\tau}e^{-\tau} = 1$$
- The second equation,
  $$e^{\tau} = \frac{\sin(\nu)}{\nu}$$
  has an increasing number of solutions as τ grows more and more negative.

- As $\tau \to -\infty$, the number of solutions becomes infinite
  and are given by the zeros of $\sin(\nu)$
- At $\tau \to -\infty$, the solutions of the other equation converge to the zeros of $\cos(\nu)$
- Thus, the solutions of

$$\begin{aligned} e^{\tau}(1-\tau) &= \cos(\nu) \\ e^{\tau} &= \frac{\sin(\nu)}{\nu} \end{aligned}$$

  may not necessarily occur together for $\tau < 0$, as they must.

Poles at s=0

We know from the foregoing that the denominator of the cone reflectance has at least one root at $s = 0$. We now investigate the "dc behavior" more thoroughly.

- A hasty analysis based on the reflection and transmission filters (see figure) might conclude that the reflectance of the conical cap converges to -1 at dc, since $R(0) = -1$ and $T(0) = 0$. However, this is incorrect.

- Instead, it is necessary to take the limit as $\omega \to 0$ of the complete conical cap reflectance $R_J(s)$:

$$R_J(s) = \frac{1 - e^{-2s} - 2se^{-2s}}{2s - 1 + e^{-2s}}$$

We already discovered a root at $s = 0$ in the denominator in the context of the preceding stability proof. However, note that the numerator also goes to zero at $s = 0$. This indicates a pole-zero cancellation at dc.

- To find the reflectance at dc, we may use L'Hospital's rule to obtain

$$R_J(0) = \lim_{s\to 0} \frac{N'(s)}{D'(s)} = \lim_{s\to 0} \frac{4se^{-2s}}{2 - 2e^{-2s}}$$

and once again the limit is an indeterminate $0/0$ form.

- We apply L'Hospital's rule again to obtain

$$R_J(0) = \lim_{s\to 0} \frac{N''(s)}{D''(s)} = \lim_{s\to 0} \frac{(4 - 8s)e^{-2s}}{4e^{-2s}} = +1$$

Thus, two poles and zeros cancel at dc, and the dc reflectance is $+1$, not -1 as an analysis based only on the scattering filters would indicate.

- From a physical point of view, it makes more sense that the cone should "look like" a simple rigid termination of the cylinder at dc, since its length becomes vanishingly small compared with the wavelength in the limit.

- Another method of showing this result is to form a Taylor series expansion of the numerator and denominator:

$$
\begin{aligned}
N(s) &= 2s^2 - \frac{8s^3}{3} + 2s^4 + \cdots \\
D(s) &= 2s^2 - \frac{4s^3}{3} + \frac{2s^4}{3} + \cdots
\end{aligned}
$$

  Both series begin with the term $2s^2$ which means both the numerator and denominator have two roots at $s = 0$. Hence, again the conclusion is two pole-zero cancellations at dc.

- The series for the conical cap reflectance is

$$
R_J(s) = 1 - \frac{2s}{3} + \frac{2s^2}{9} - \frac{4s^3}{135} - \frac{2s^4}{405} + \cdots
$$

  which approaches $+1$ as $s \to 0$.