Chapter 1

Partial Differential Equations in Musical Acoustics

At one level, the study of musical acoustics is the study of physics, and in particular, continuum mechanics. Taking this point of view, a musical instrument, which is always a solid object of a given geometry encompassing and surrounded by air, and driven by some exciting mechanism, can be fully described by sets *partial differential equations*, which are statements of basic conservation laws, accompanied by appropriate boundary conditions. In theory, then, a computer simulation which calculates the radiated sound field from an instrument can be performed, in much the same way the the flow and pressure fields around an airfoil are calculated, by brute force. This is, however, easier said than done, and, even if done, prohibitively expensive from a computational point of view[1].

The game, then, if one is interested in musical sound synthesis in something approaching real time (and this is becoming a possibility for some of the algorithms presented in the next chapter) is to make as many simplifications as possibility without sacrificing the salient qualities of the instrument vibration itself. Among these simplifications will be assumptions of linearity and reduced dimensionality, which are often well-justified and lead to simplified PDEs—as it turns out, this set of useful PDEs for musical sound synthesis is in fact fairly small. That being said, we will not spend much time justifying these equations from a physical point of view, but will simply take these very commonly-encountered forms as a point of departure for a discussion of numerical methods (see Chapter **??**). A reader interested in the physical underpinnings of these forms is referred to one of the many comprehensive texts on the subject [].

For some unknown reason, electrical engineers have often been entrusted with the solemn duty of squeezing musical sounds out of computers. But PDEs

[1]

and numerical methods, while familiar to mechanical engineers and applied mathematicians, are not usually covered in an electrical engineering curriculum. This is unfortunate, because the basic concepts, techniques and tools will be very familiar to anyone with a knowledge of digital or analog filter design, Fourier and Laplace transforms, DFTs and z-transforms, linear system theory, and basic complex analysis—in other words, an electrical engineer! Those who have studied more advanced topics such as linear system theory, matrix analysis, Fourier optics and digital image processing will find their effort paying off unexpected dividends when it comes to the material we are about to present, in this chapter and the next.

1.1 PDEs in One Spatial Dimension

For many of the key components of musical instruments (namely strings, acoustic tubes and thin bars), vibration wavelengths are long compared to all but one spatial dimension (though sometimes, as for a brass instrument, this dimension is curved). In these cases, it is advantageous to reduce the PDE description of the motion to an equation in one spatial variable x and a time variable t. All the PDEs which arise are variations or elaborations on a couple of key forms.

1.1.1 The Wave Equation

The single most important PDE in all of musical acoustics is the *one-dimensional wave equation*, which is written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

Here, $t \in \mathbb{R}$ is a time variable, $x \in \mathbb{R}$ is a spatial variable, and c is known as the *wave speed.* $u(x,t)$ is the unknown or dependent variable which we wish to solve for, and the key characteristic of (**??**) is that it possesses wave-like solutions, as we will see shortly.

Equation (1.1) is a useful first approximation to the vibration of many systems which appear frequently in musical acoustics. For instance, $u(x,t)$ could be the transverse displacement or velocity, or force of a violin or piano string under tension, or the longitudinal displacement of an ideal stiff bar, or the longitudinal volume velocity or pressure in a thin, straight cylindrical acoustic tube such as certain woodwind and brass instrument, and to a rougher approximation, the human vocal tract. (It also can be used to describe one-dimensional electromagnetic waves, or voltage and current waves on a transmission line, which are perhaps more familiar to electrical engineers.) The value of c depends, always, on two properties of (or conditions on) the material under consideration, assumed constant. One of these constants will always be the *mass density* ρ, and the other will always be the *stiffness*. It is useful to recall, from the treatment of coupled mass-spring systems in elementary physics, that these are the are the two "dual" properties that a system must possess in order for oscillations to occur. For a string, $c=\sqrt{T/\rho}$, where T is the tension applied to the string, in Newtons (giving rise to stiffness). For a bar, $c=\sqrt{E/\rho}$, where E is Young's modulus, a property of the material from which the bar is constructed. For an acoustic tube, $c=\sqrt{K/\rho}$, where K is the bulk modulus of air. Note that speed always increases as stiffness is increases (i.e., the medium reacts faster to perturbations), and decreases as mass is increased (i.e., the medium becomes more inertial).

System Interpretation

It is important to keep in mind that the one-dimensional wave equation, second order in both the time and space variables, is always derived from a first-order set of equations which can be related more directly to definitions of basic quantities and conservation equations. For the string, the generating system is

$$\rho\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x} \tag{1.2a}$$

$$\frac{\partial f}{\partial t} = T\frac{\partial v}{\partial x} \tag{1.2b}$$

for u, the transverse velocity of the string, and $f(x,t)$, the vertical force. Taking a time derivative of the first equation and a space derivative of the second, and eliminating terms gives (1.1). A wave equation in force f follows by performing the dual set of manipulations. It is sometimes useful to have access to this more fundamental form, in case certain variations are to be introduced. If, for instance, our string were to have a varying mass density $\rho(x)$, it is not clear whether a form such as (1.1) exists; indeed, the first-order system tells us the answer (see Problem **??**).

The D'Alembert Solution

Though in general, it is impossible to obtain explicit solutions for PDEs (and indeed, methods for obtaining approximate numerical solutions are the subject of this chapter), the one-dimensional wave equation does in fact yield explicit solutions. It is useful to first rewrite the wave equation in the form

$$\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)u=0 \tag{1.3}$$

Then, defining the new variables w and v by

$$w=t+cx \qquad\qquad v=t-cx \tag{1.4}$$

the wave equation can be rewritten as

$$\frac{\partial^2 u}{\partial w\partial v}=0 \tag{1.5}$$

It should be clear that integrating twice, once with respect to w, and then with respect to v introduces two arbitrary functions in w and v,

$$u(w, v) = f(w) + g(v) \tag{1.6}$$

and thus, transforming back to the variables x and t,

$$u(x, t) = f(x + ct) + g(x - ct) \tag{1.7}$$

A few comments are in order at this point. First, note that $u(x, t)$ is a sum of two functional forms, or *waves* of arbitrary shape. We are justified in calling them waves, because one moves to the right at a speed c, the other to the left at the same speed; the shapes travel without any deformation.

The above traveling wave decomposition forms the basis for digital waveguide methods.

1.1.2 The Wave Equation with Loss

The above model of wave propagation is very crude; waves travel unattenuated for all time. Real physical systems always exhibit some form of loss, which may be internal to the medium itself (through, e.g., viscoelastic or thermal effects), or perhaps through transfer to the surroundings (i.e., radiation). A simple "resistive" model of loss can be obtained by adding a first time derivative term to (1.1), giving

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - 2b_1 \frac{\partial u}{\partial t} \tag{1.8}$$

The loss parameter b_1 is assumed to be positive here; for $b_1 = 0$, the wave equation is recovered.

In this case, there is no traveling wave solution; the resulting motion of the medium is much more complicated. In fact, there is no longer a single "speed" associated with the motion of the medium, though it is possible to show, at least, that disturbances travel at bounded speeds—see §**??**. (It is possible, however, to modify this equation slightly so that traveling wave type solutions again reappear—see Problem **??**. The lossy traveling wave model is also quite useful from the point of view of digital waveguide synthesis.)

1.1.3 The Ideal Beam Equation

A very different, but nonetheless physical type of motion is governed by the *Euler-Bernoulli* or *ideal beam* equation

$$\frac{\partial^2 u}{\partial t^2} = -\kappa^2 \frac{\partial^4 u}{\partial x^4} \tag{1.9}$$

Here, u is the transverse displacement of the beam, and κ is a stiffness parameter; this equation models transverse vibrations on a thin stiff bar, where the restoring force is supplied by the rigidity of the bar, instead of tension, as in a string.

In contrast to the wave equation, neither the lossy wave equation nor the ideal bar equation possesses a simple traveling wave solution. We will return to the analysis of these PDEs in detail in the following sections.

1.2 Frequency domain analysis

Many useful PDE models in musical acoustics have the special property that they are invariant under shifts in space or time; that is, the system behaves the same "here and now" as it did "then and over there." What is more, they are linear: if u_1 and u_2 are solutions, then so is any linear combination $\alpha u_1 + \beta u_2$. This linear shift-invariant property is exhibited by systems (1.1), (1.8) and (1.9). (If, in the wave equation, c were dependent on x or t, the equation would be linear but not shift-invariant, and if c were dependent on u, it would not be linear.) As it turns out, this property leads to a much-simplified analysis in the frequency domain.

1.2.1 Spatial Fourier Transforms

Let us first define the spatial Fourier transform of a function $u(x,t)$ by

$$\hat{u}(\beta,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-i\beta x}dx \tag{1.10}$$

Here β, assumed real, is the *wave number*, corresponding to a component of wavelength $2\pi/\beta$. The transform is inverted by

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\beta,t)e^{i\beta x}d\beta \tag{1.11}$$

Parseval's relation holds for such a transform pair, denoted here as $u \longleftrightarrow \hat{u}$, i.e.,

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx = \int_{-\infty}^{\infty} |\hat{u}(\beta,t)|^2 d\beta \tag{1.12}$$

Transforms of Spatial Differential Operators

Taking the qth spatial derivative of the inverse transform definition (1.11) gives

$$\frac{\partial^q u}{\partial x^q} = \int_{-\infty}^{\infty} (i\beta)^q \hat{u}(\beta,t)e^{i\beta x}d\beta \tag{1.13}$$

provided that $u(x,t)$ is sufficiently differentiable. In other words,

$$\frac{\partial^q u}{\partial x^q} \longleftrightarrow (i\beta)^q \hat{u} \tag{1.14}$$

As is the case for the analysis of electric circuits, or lumped mechanical systems, the interest in working in the transform domain derives precisely from the identification of differential operators with algebraic operators.

1.2.2 Fourier Transforms of PDEs

Suppose we now take the Fourier transform of the wave equation (1.1). Keeping in mind the transforming rule (1.14) for spatial derivatives, we obtain

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\beta^2\hat{u} \tag{1.15}$$

This is no more than a second-order linear *ordinary differential equation* (ODE) in the variable $\hat{u}(\beta, t)$; in fact, for a fixed value of β, it describes a *simple harmonic oscillator* (SHO), of angular frequency $\omega = c\beta$.

The transform of the lossy wave equation is

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -c^2\beta^2\hat{u} + 2b_1\frac{\partial \hat{u}}{\partial t} \tag{1.16}$$

which is, again, a second-order ODE, corresponding, now, to a *damped* harmonic oscillator.

The transform of the ideal beam equation is

$$\frac{\partial^2 \hat{u}}{\partial t^2} = -\kappa^2\beta^4\hat{u} \tag{1.17}$$

which is again a SHO; the frequency, however, is now given by $\omega = \kappa\beta^2$.

1.2.3 Characteristic Polynomials

All the Fourier-transformed PDEs discussed above are of the form

$$\frac{\partial^2 \hat{u}}{\partial t^2} + 2p(\beta)\frac{\partial \hat{u}}{\partial t} + q(\beta) = 0 \tag{1.18}$$

for some polynomial functions $p(\beta)$ and $q(\beta)$ (here, real-valued). As mentioned previously, these are simply linear, constant-coefficient second-order ODEs, as can easily be seen by suppressing the explicit dependence on β. The familiar general solution will be of the form

$$\hat{u}(\beta, t) = a_+(\beta)e^{s_+(\beta)t} + a_-(\beta)e^{s_-(\beta)t} \tag{1.19}$$

for two numbers $s_+(\beta)$ and $s_-(\beta)$ which are the solutions (assumed distinct, for the moment) to the characteristic polynomial equation

$$s^2 + 2p(\beta)s + q(\beta) = 0 \tag{1.20}$$

which is obtained by inserting a trial solution of the form $\hat{u} = e^{st}$ into (1.18). The roots can be written explicitly as

$$s_\pm(\beta) = -p(\beta) + \sqrt{p^2(\beta) - q(\beta)} \tag{1.21}$$

As seen from (1.19), the real parts of the roots $s_\pm$ will control the growth or decay of the solution, and the imaginary parts affect the phase only. We will write, in general,

$$s_\pm(\beta) = \sigma_\pm(\beta) + i\omega_\pm(\beta) \tag{1.22}$$

The two arbitrary functions $a_+(\beta)$ and $a_-(\beta)$ can be written in terms of the Fourier transforms of the initial data as

$$a_\pm(\beta) = \pm\frac{s_\mp(\beta)\hat{u}(\beta,0) - \frac{\partial\hat{u}(\beta,0)}{\partial t}}{s_- - s_+} \tag{1.23}$$

Frequency domain analysis is useful in the context of PDE analysis for exactly the same reasons that they are used in the analysis of linear and shift-invariant lumped mechanical and electrical systems—differential equations are transformed into algebraic equations (generally polynomial). The analysis can thus be simplified, but only to a point. As it turns out, the algebraic equations are nonlinear, and often, the stability analysis for a PDE system is reduced to an attempt to say specific things about the roots of these algebraic equations.

1.2.4 Well-posedness

The roots of the characteristic equation play an important role in determining whether the system in question behaves physically; after all, we can write down any PDE we want, but there is no guarantee that it describes the behavior of a real system. One property that an initial-value problem should have is known as *well-posedness*; though we can't fully define this property here, it can be described as a requirement that the solution of the PDE grow no faster than exponentially. This condition has implications for the existence and uniqueness of solutions; it should be true that the solution to a differential equation varies in a reasonable way with the initial values provided to it. Of course, for musical instrument modeling, this condition is somewhat overly technical, and is fact not strong enough; we would generally want a condition that solutions be damped, or at least non-increasing (that is, acoustic musical instruments are *passive*). But well-posedness is a good starting point for such a discussion.

First, consider the Fourier transformed solution (**??**), and, to simplify matters, suppose that we have chosen the initial conditions such that $a_-(\beta) = 0$; this then implies that $a_+(\beta) = \hat{u}(\beta,0)$ (as can be easily checked). We then have

$$\hat{u}(\beta,t) = \hat{u}(\beta,0)e^{s_+(\beta)t} = \hat{u}(\beta,0)e^{\sigma_+(\beta)t}e^{i\omega_+(\beta)t} \tag{1.24}$$

This simplifying assumption essentially reduces the problem to first-order. Suppose now that there is a number σ_+^* such that $\sigma_+(\beta) \leq \sigma_+^*$. Then, we have

$$|\hat{u}(\beta,t)| = |\hat{u}(\beta,0)|e^{\sigma_+(\beta)t} \leq |\hat{u}(\beta,0)|e^{\sigma_+^* t} \tag{1.25}$$

and, squaring and integrating over β,

$$\int_{-\infty}^{\infty} |\hat{u}(\beta,t)|^2 d\beta \leq e^{\sigma_+^* t}\int_{-\infty}^{\infty} |\hat{u}(\beta,0)|^2 d\beta \tag{1.26}$$

Parseval's relation gives the bound

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \leq e^{\sigma_+^* t}\int_{-\infty}^{\infty} |u(x,0)|^2 dx \tag{1.27}$$

so the size of the solution (in an L^2 norm) is bounded in terms of the size of the initial condition. It should be clear that a similar bound on $\sigma_-(\beta)$ is also necessary for well-posedness. Actually, for musical applications, we would want the stronger condition that

$$\sigma_\pm(\beta) \leq \sigma_\pm^* \leq 0 \tag{1.28}$$

which corresponds, roughly speaking, to passivity—the "energy" of the system at any time $t \geq 0$ is *less than* that it had at $t = 0$.

The above analysis is very crude, in several ways, but it does give a general indication of the utility of spectral analysis; it is a direct extension of Laplace-type analysis of lumped systems, where stability is typically characterized in terms of system pole locations s. Here, the pole locations s are functions $s(\beta)$ of wavenumber, and what requirements we make for well-posedness, or passivity need to hold over *all* wavenumbers. What have we neglected? First, we have simplified the analysis of the second-order problem by eliminating one of the two solutions; if both are retained, the analysis is more complex, but a boundedness condition on the two roots of the characteristic equation is necessary and sufficient for well-posedness. Second, we have examined a pure initial value problem (i.e., the spatial domain is unbounded); a complete analysis of the well-posedness of solutions to the equation must take such conditions into account.

1.2.5 Dispersion Relations and Phase Velocity

Besides gross information regarding well-posedness, the characteristic polynomial for a PDE also supplies us with finer information, from which we can say a good deal about how solutions evolve.

First, note that another way of obtaining the dispersion relation is by simply *assuming* a "plane-wave" solution of the form

$$u(x,t) = e^{st+i\beta x} \tag{1.29}$$

Though this is a short-cut, it can be justified by an appeal to Fourier and Laplace theory—any solution can be expanded in terms of such exponentials, and, by linearity, it is sufficient to examine any single component in isolation.

Let's look again at the solutions to the

Let us now apply these transforms to the three equations discussed in §**??**, beginning with the wave equation (**??**), first under steady state conditions, and assuming no spatial boundaries. After Fourier transforming, we obtain

$$\frac{\partial^2 \hat{y}}{\partial t^2} = -c^2\beta^2\hat{y} \tag{1.30}$$

This is no more than a second-order linear *ordinary differential equation* (ODE) in the variable $\hat{y}(\beta,t)$; in fact, it is a *harmonic oscillator*. Now, applying the Laplace transorm, we obtain

$$s^2\tilde{\hat{y}} = -c^2\beta^2\tilde{\hat{y}} \Rightarrow \underbrace{(s^2 + c^2\beta^2)}_{P(s,\beta)}\tilde{\hat{y}} = 0 \tag{1.31}$$

The quantity $P(s,\beta)$ defined above is known as the *symbol* for (**??**).

It is worth noting that we could have obtained the same equation by, instead of Fourier and Laplace transforming, simply *assuming* a solution of the form

$$y(x,t) = e^{st+j\beta x} \tag{1.32}$$

Thus, we can think of (**??**) as describing the behavior of a solution to the wave equation at the single frequency pair (s,β). In particular, as we shall see shortly, (**??**) tells us exactly what the relationship between s and β must be for such a solution to exist.

We can see that unless $P(s,\beta) = 0$, then this equation has only the trivial solution $\tilde{\hat{y}} = 0$. $P(s,\beta) = 0$ defines what is known as the *characteristic equation* for the wave equation, namely

$$s^2 = -c^2\beta^2 \tag{1.33}$$

Obviously, since β is real, then s must be purely imaginary, and writing $s = j\omega$, we obtain the following *dispersion relations*:

$$\omega = \pm c\beta \tag{1.34}$$

It is important that there are *two* dispersion relations; as we will see shortly, these can be associated directly with the left- and right-traveling waves mentioned in §**??**. Notice also that because $s = j\omega$ is purely imaginary, a solution $y(x,t) = e^{j\omega t+j\beta x}$ is "undamped", i.e., $|y(x,t)| = 1$. A purely imaginary s reflects the *losslessness* of a system such as the wave equation.

For the bar equation, the analysis is very similar; after Fourier and Laplace transforming (or, equivalently, assuming a solution $y = e^{st+j\beta x}$), we obtain the characteristic equation

$$s^2 + \kappa^2\beta^4 = 0 \tag{1.35}$$

from which it is obvious that s must again purely imaginary for a solution to exist. We obtain the dispersion relations

$$\omega = \pm\kappa\beta^2 \tag{1.36}$$

For the lossy wave equation, the situation is a bit different. Applying the same analysis as before, we get the characteristic equation

$$s^2 + 2b_1 s + c^2\beta^2 = 0 \tag{1.37}$$

Here, it is no longer true that s is purely imaginary. In fact, the solutions are

$$s_{\pm} = -b_1 \pm \sqrt{b_1^2 - c^2\beta^2} \tag{1.38}$$

Supposing that $s = \sigma + j\omega$, there are now two different regimes, depending on the value of the wavenumber β. If $|\beta| \geq |b_1|/c$, then

$$\sigma = -b_1 \qquad \omega = \pm\sqrt{c^2\beta^2 - b_1^2} \tag{1.39}$$

This is the regime of normal damped wave propagation. We note two things here. First,

$$\omega \approx \pm c\beta \qquad \text{for} |\beta| >> |b_1|/c \tag{1.40}$$

or, in other words, if damping is small, the dispersion relation is not appreciably different from what it is in the lossless case. Second, for a solution $y(x,t) = e^{st+j\beta x}$, we have

$$|y(x,t)| = |e^{st+j\beta x}| = |e^{st}| = |e^{-b_1 t}| \tag{1.41}$$

Thus for $b_1 > 0$, the solution is damped, *independently of frequency or wavenumber*. Such loss is called, naturally, *frequency independent*. (For an examination of a system exhibiting frequency-dependent loss, turn to §**??**.)

The other propagation regime, i.e., for $|\beta| < |b_1|/c$, we have

$$\sigma = -b_1 \pm \sqrt{b_1^2 - c^2\beta^2} \qquad \omega = 0 \tag{1.42}$$

Thus below a certain "cutoff" wavenumber $|\beta_c| = |b_1|/c$, we have non-oscillatory solutions. If

1.2.6 Phase Velocity

Let's return to the dispersion relations for the wave equation,

$$\omega(\beta) = \pm c\beta \tag{1.43}$$

As we mentioned in the last section, the dispersion relation can be obtained by inserting a solution of the form $y(x,t) = e^{st+j\beta x}$ into the PDE itself, and setting the symbol $P(s,\beta) = 0$. Now that we have obtained these solutions, let us go back and look at what these test solutions look like. We have

$$y(x,t) = e^{j\omega(\beta)t+j\beta x} = e^{j\beta(x\pm ct)} \tag{1.44}$$

It should be obvious that these solutions are nothing more than a pair of traveling waves, of the form of (**??**). They travel at speed c.

What about the ideal bar equation? The dispersion relations are now

$$\omega(\beta) = \pm\kappa\beta^2 \tag{1.45}$$

and our test solutions $y(x,t) = e^{st+j\beta x}$ will thus be

$$y(x,t) = e^{j\omega(\beta)t+j\beta x} = e^{j\beta(x\pm c_\phi(\beta)t)} \tag{1.46}$$

where we have defined the *phase velocity* $c_\phi(\beta)$ by

$$c_\phi(\beta) = |\frac{\omega(\beta)}{\beta}| = |\kappa\beta| \tag{1.47}$$

The phase velocity has the interpretation of a propagation speed which is dependant on the wavenumber; for the wave equation, the phase velocity is simply

$c_\phi = c$, and is independent of the wavenumber. The implication is that if a bar is deformed an then released at time $t = 0$, the deformation will propagate, but the various spatial frequency components of the deformation travel at different speeds, and the deformation's shape must therefore become distorted or *disperse* as time progresses. Thus wave propagation in an ideal bar is called *dispersive.* As we can see, the phase velocity increases linearly with the wavenumber, so that short wavelengths travel faster than long wavelengths.

1.3 Boundary Conditions

So far, we have looked at PDEs defined over the domain $x \in \mathbb{R}$; this is obviously not the case for any musical instrument

1.3.1 Boundary Conditions for the Wave Equation

1.3.2 Boundary Conditions for the Ideal Beam Equation

1.4 PDEs in higher dimensions

Certain musical instruments employ vibrating elements which cannot be reduced to a simple description in one spatial dimension. We will focus, in this section, on PDEs in two spatial Cartesian coordinates, x and y. Drum membranes, and stiff plates such as cymbals and gongs, as well as the piano soundboard are the most important examples of objects requiring such a treatment; . Simple solutions to two-dimensional PDEs are even more hard to come by than in 1D, both due to the inherent properties of higher-dimensional equations (the 2D wave equation, for instance, possesses no simple traveling-wave solution), and the many new geometrical possibilities.

1.4.1 The 2D Wave Equation and the Ideal Plate Equation

The 1D wave equation (**??**) can be generalized in a straightforward way to 2D:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \qquad \nabla^2 \triangleq \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{1.48}$$

c is again the wave speed (we will justify calling it this in the next section), and $u(x, y, t)$ is the dependent variable to be solved for. If u is the transverse displacement of a membrane of mass density ρ and under uniform applied tension T, the wave speed is given by $c = \sqrt{T/\rho}$, just as for the ideal string.

It is important to note the spatial shift-invariance of this equation, which allows, as in 1D, convenient spectral analysis techniques to be employed. This shift-invariance is, however, contingent on the use of Cartesian coordinates, and is otherwise lost—this is worth keeping in mind if the geometry of the system to be modeled does not conform to rectilinear coordinates (and indeed, in musical applications, few do). Even in Cartesian coordinates, however, there is no change of variables which can be employed to give a D'Alembert type traveling-wave solution; there is no simplified description of the motion of such a system.

Similarly, the ideal beam equation may be generalized to 2D as

$$\frac{\partial^2 u}{\partial t^2} = -\kappa^2\nabla^4 u \qquad\qquad \nabla^4 \triangleq \nabla^2\nabla^2 \tag{1.49}$$

This equation describes the transverse motion of a thin flat plate. The stiffness parameter κ is given by

$$\kappa = \tag{1.50}$$

where ρ is the mass density, h is the thickness, E is Young's modulus, and ν is another material parameter known as Poisson's ratio.

Both the wave equation and the plate equation are second order in the time variable, and thus require two initial conditions, $u(x, y, 0)$ and $\frac{\partial u(x,y,0)}{\partial t}$.

1.4.2 Spectral Analysis in 2D

As we mentioned earlier, both the 2D wave equation and the plate equation are linear and shift-invariant, and are thus amenable to Fourier analysis.

The Spatial Fourier Transform in 2D

The Fourier Transform in 2D of a function $u(x, y, t)$ is defined by

$$\hat{u}(\beta_x, \beta_y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x, y, t)e^{-i(\beta_x x+\beta_y y)}dxdy \tag{1.51}$$

The two transform variables, β_x and β_y both have the dimension of a wavenumber, and are best interpreted as the components of a vector wavenumber $\beta = [\beta_x, \beta_y]$, which defines the *direction* of a single sinusoidal component of the form $e^{-i(\beta_x x+\beta_y y)}$. The function $\hat{u}(\beta_x, \beta_y, t)$ is thus an expansion of $u(x, y, t)$ into a sum of such directional sinusoids. The wavelength of any single such component will be $2\pi/|\beta|$. The transform can be inverted by

$$u(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\beta_x, \beta_y, t)e^{i(\beta_x x+\beta_y y)}d\beta_x\beta_y \tag{1.52}$$

Parseval's relation holds for the transform pair $u \longleftrightarrow \hat{u}$ in 2D as well.

Spatial Differential Operators in the Transform Domain

The operators ∇^2 and ∇^4 transform according to

$$\nabla^2 \quad \longleftrightarrow \quad -|\beta|^2 \tag{1.53}$$

$$\nabla^4 \quad \longleftrightarrow \quad |\beta|^4 \tag{1.54}$$

Notice that the transforms of these operators depend only on the magnitude of the vector wavenumber, and not its individual components. That is, they depend only on the wavelength and not direction. When such operators appear in a PDE, they immediately signal *isotropic* behavior (i.e., motion which is independent of direction). Isotropy is a characteristic peculiar to systems in more than one spatial dimension; while the wave equation and the plate equation, as we have presented them, possess this properties, there are important examples of musical instrument components which are not isotropic (the wood grain effects in a piano soundboard, for instance, lead to such behavior).

1.4.3

1.5 Problems

Consider the first-order system (1.2), describing the transverse motion of a vibrating string, in a single plane of polarization. Suppose the linear mass density ρ is a smooth function of x, and derive a single second-order equation in v. Derive another in f. What are possible sources of spatial variation in (a) longitudinal vibration in bars and (b) acoustic tubes? Give examples of each type of variation in 1D musical acoustics.

As we have seen, when a loss term is introduced to the wave equation, in (**??**), wave propagation is dispersive. This is not always the case, however. Consider the same equation with an extra term,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - 2b_1\frac{\partial u}{\partial t} - b_1^2 u \tag{1.55}$$

(a) Find the characteristic equation for this PDE, and solve for the roots $s_\pm(\beta)$. What are the real and imaginary parts of these roots? What can you deduce about the loss and dispersion characteristics of solutions to this equation?

(b) Introduce a new variable $w(x,t) = u(x,t)e^{b_1 t}$, and rewrite the PDE above in terms of w. Find a general solution $w(x,t)$ in terms of traveling waves, and then write the general solution $u(x,t)$. How would you characterize this solution?

The equation above corresponds to “distortionless” wave propagation on an electrical transmission line. A discrete-time and space analogue exists; see Problem **??** in the next chapter.

Consider the system defined by

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \kappa^2\frac{\partial^4 u}{\partial x^4} + 2b_1\frac{\partial u}{\partial t} - 2b_3\frac{\partial^3 u}{\partial t^3} \tag{1.56}$$

where $b_1 \geq 0$, $b_3 \neq 0$.

(a) Write down the characteristic polynomial for this PDE, neglecting initial and boundary conditions (it will be an equation in s and β).

(b) It is tedious to solve for the roots of this polynomial (in s) directly. How many roots $s(\beta)$ will there be? Given that the polynomial coefficients are real, what can you say about the roots? What is the behavior of the roots of this equation in the limit as β becomes large, for $b_3 > 0$, and $b_3 < 0$? Is the system well-posed?

(c) Optional: numerically solve for the roots of this equation over $0 \leq \beta \leq 100$. Characterize the real and imaginary parts of the roots (are they positive, negative, dependent on β?).

(d) Suppose the term $2b_3\frac{\partial^3 u}{\partial t^3}$ is replaced by $2b_2\frac{\partial^3 u}{\partial t \partial x^2}$, for some $b_2 \geq 0$. Write down a characteristic polynomial for this system, and solve for the roots $s(\beta)$ explicitly. Is this system well-posed?