FM Spectra Derived

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Music 320 Extensions

Frequency Modulation (FM)

A general formula for *Frequency Modulation* (FM) used in sound synthesis is

$$\begin{aligned} y(t) &= A_c \cos\{\omega_c t + [\phi_c + A_m \cos(\omega_m t + \phi_m)]\} \quad \text{(\textit{phase modulation})} \\ &= A_c \cos\{[\omega_c - (A_m \omega_m) \sin(\omega_m t + \phi_m)]t + \phi_c\} \quad \text{(\textit{frequency modulation})} \end{aligned}$$

- A_c is the *carrier amplitude*
- A_m is the *modulator amplitude*
- $\omega_c = 2\pi f_c$ is the *carrier frequency* in radians per second
- $\omega_m = 2\pi f_m$ is the *modulation frequency*
- ϕ_m is the *modulation phase* in radians
- ϕ_c is the *carrier phase*

Reference:

`https://ccrma.stanford.edu/~jos/mdft/Sinusoidal_Frequency_Modulation_FM.html`

It is well known that FM spectra involve *Bessel coefficients*.

How do we derive this?

[**Prerequisite:** *Music 320* or equivalent understanding of complex variables up to Euler's Identity and Taylor series]

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- Bessel Coefficients
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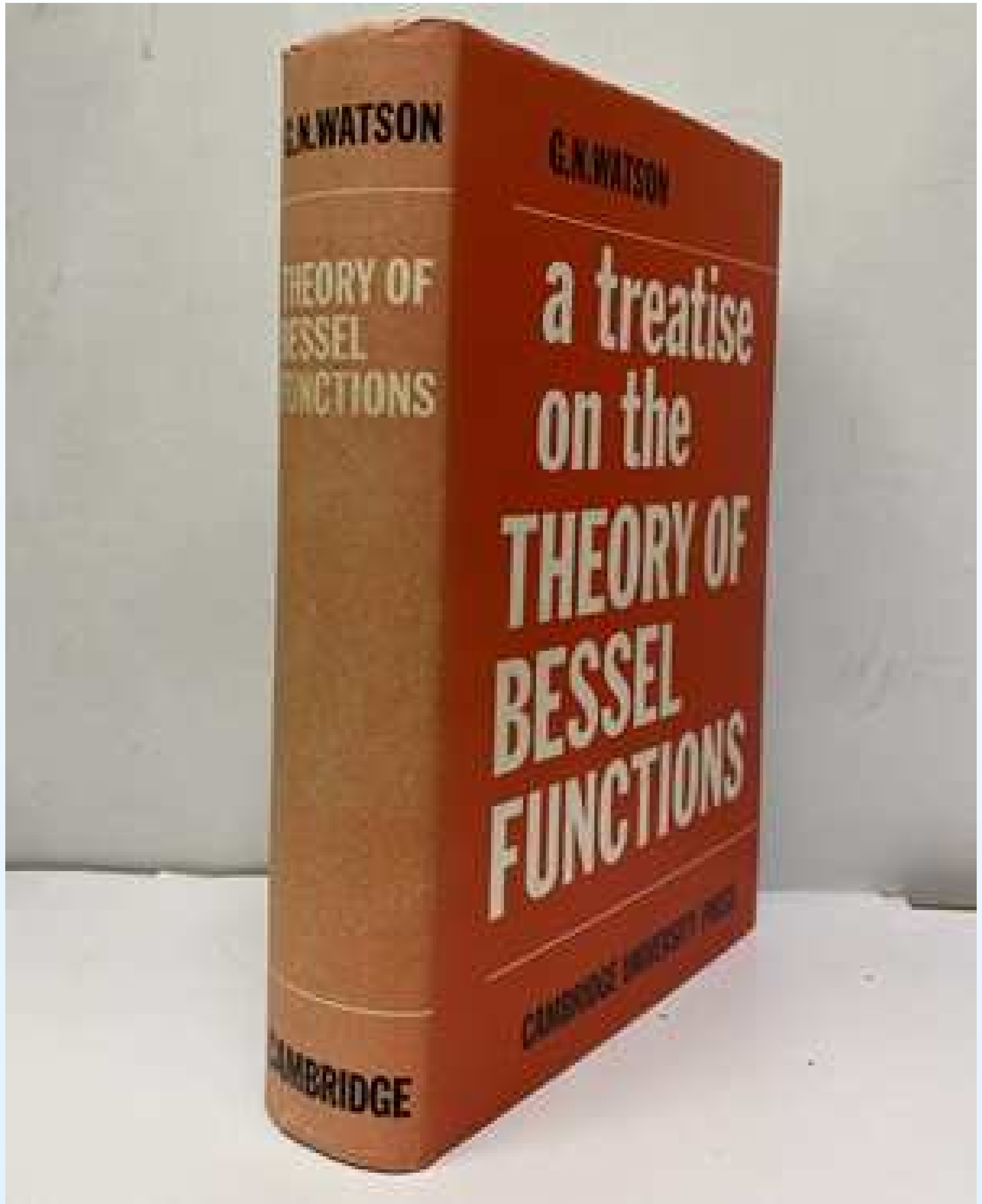
Claude Flails

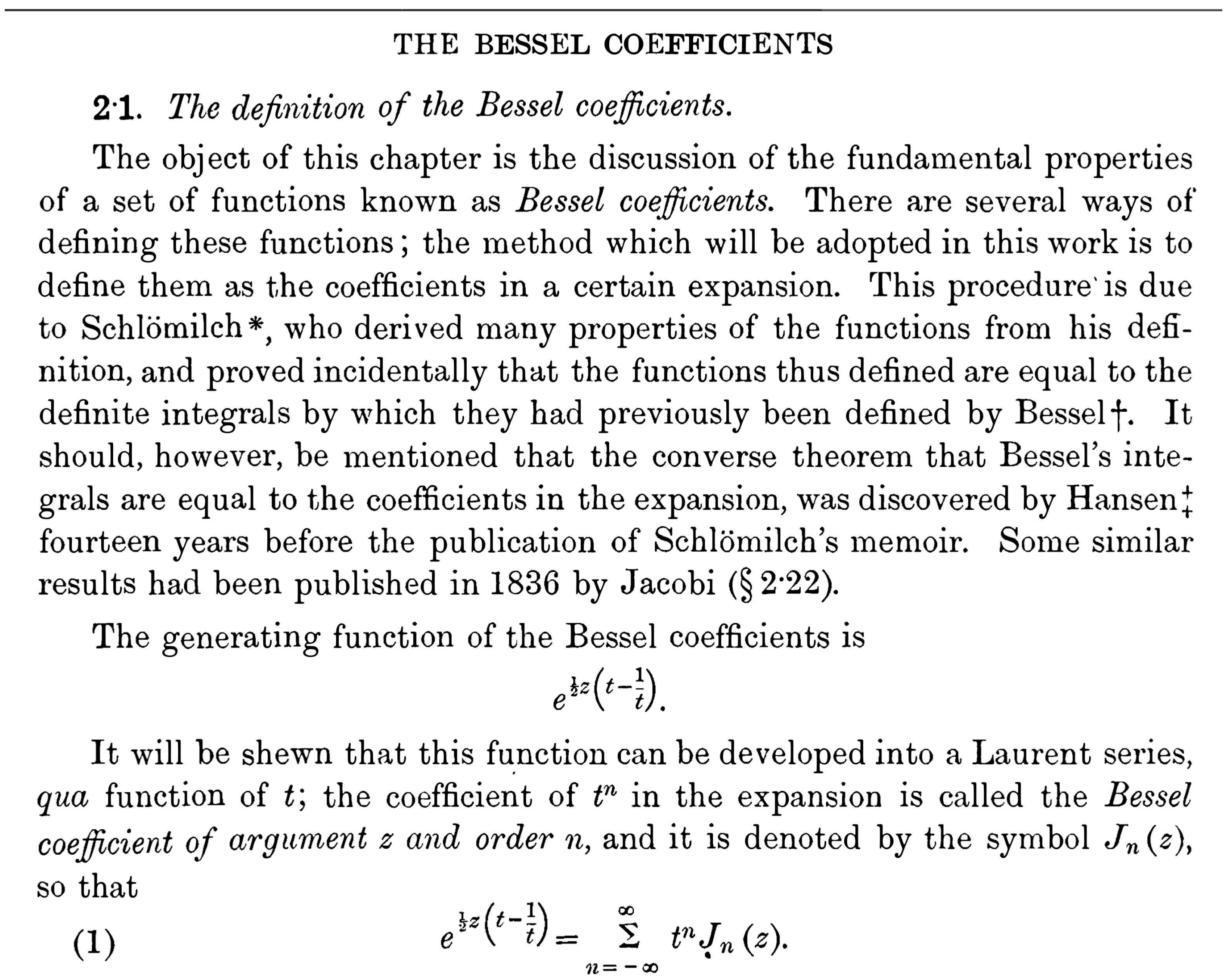
ChatGPT and Gemini

All Three

Watson



Bessel Coefficients Defined — G. N. Watson (1944, p. 14)



THE BESSEL COEFFICIENTS

2·1. *The definition of the Bessel coefficients.*

The object of this chapter is the discussion of the fundamental properties of a set of functions known as *Bessel coefficients*. There are several ways of defining these functions; the method which will be adopted in this work is to define them as the coefficients in a certain expansion. This procedure is due to Schlömilch*, who derived many properties of the functions from his definition, and proved incidentally that the functions thus defined are equal to the definite integrals by which they had previously been defined by Bessel†. It should, however, be mentioned that the converse theorem that Bessel's integrals are equal to the coefficients in the expansion, was discovered by Hansen‡ fourteen years before the publication of Schlömilch's memoir. Some similar results had been published in 1836 by Jacobi (§ 2·22).

The generating function of the Bessel coefficients is

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)}.$$

It will be shewn that this function can be developed into a Laurent series, *qua* function of t; the coefficient of t^n in the expansion is called the *Bessel coefficient of argument z and order n*, and it is denoted by the symbol $J_n(z)$, so that

$$e^{\frac{1}{2}z\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(z). \tag{1}$$

The Bessel Coefficients

In **A Treatise on the Theory of Bessel Functions** by G. N. Watson (1944, p. 14), the *Bessel coefficients* $J_n(z)$ are *defined* by the complex series expansion (*e.g.*, about $u = 1$)

$$e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)} = \sum_{n=-\infty}^{\infty} u^n J_n(z).$$

Setting $z = \beta \in \mathbb{R}$ and $u = e^{j(\omega_m t+\phi_m)}$ gives

$$e^{j\beta\sin(\omega_m t+\phi_m)} = \sum_{n=-\infty}^{\infty} e^{jn\phi_m} e^{jn\omega_m t} J_n(\beta).$$

Multiplying both sides by $e^{j(\omega_c t+\phi_c)}$ yields

$$e^{j[\omega_c t+\phi_c+\beta\sin(\omega_m t+\phi_m)]} = \sum_{n=-\infty}^{\infty} e^{jn\phi_m} e^{j[(\omega_c+n\omega_m)t+\phi_c]} J_n(\beta).$$

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Sinusoidal Phase Modulation

So far we have

$$e^{j[\omega_c t+\phi_c+\beta\sin(\omega_m t+\phi_m)]} = \sum_{n=-\infty}^{\infty} e^{jn\phi_m} e^{j[(\omega_c+n\omega_m)t+\phi_c]} J_n(\beta).$$

Assuming $e^{jn\phi_m} = (\pm 1)^n$ (*i.e.*, $\phi_m = 0$ or π), applying Euler's identity $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, and equating real and imaginary parts gives

$$\begin{aligned}
\cos\left[\omega_c t + \phi_c \pm \beta\sin(\omega_m t)\right] &= \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(\beta)\cos[(\omega_c + n\omega_m)t + \phi_c] \\
\sin\left[\omega_c t + \phi_c \pm \beta\sin(\omega_m t)\right] &= \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(\beta)\sin[(\omega_c + n\omega_m)t + \phi_c].
\end{aligned}$$

This is sinusoidal *phase modulation* by $\Theta(t) = \pm\beta\sin(\omega_m t)$.
These results agree with Bill Schottstaedt's derivation ($+$ case) at
`https://ccrma.stanford.edu/software/snd/snd/fm.html`

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Cosinusoidal Frequency Modulation

We derived

$$\begin{aligned}\cos\left[\omega_c t + \phi_c \pm \beta \sin(\omega_m t)\right] &= \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(\beta) \cos[(\omega_c + n\omega_m)t + \phi_c],\\ \sin\left[\omega_c t + \phi_c \pm \beta \sin(\omega_m t)\right] &= \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(\beta) \sin[(\omega_c + n\omega_m)t + \phi_c].\end{aligned}$$

The *instantaneous phase* is given by

$$\Theta(t) = \omega_c t + \phi_c \pm \beta \sin(\omega_m t),$$

so that the *instantaneous frequency* is

$$\dot{\Theta}(t) = \omega_c \pm \beta\, \omega_m \cos(\omega_m t)$$

We see that sinusoidal phase-modulation by β radians corresponds to *cosinusoidal* frequency modulation by $\beta\omega_m$ radians per second.

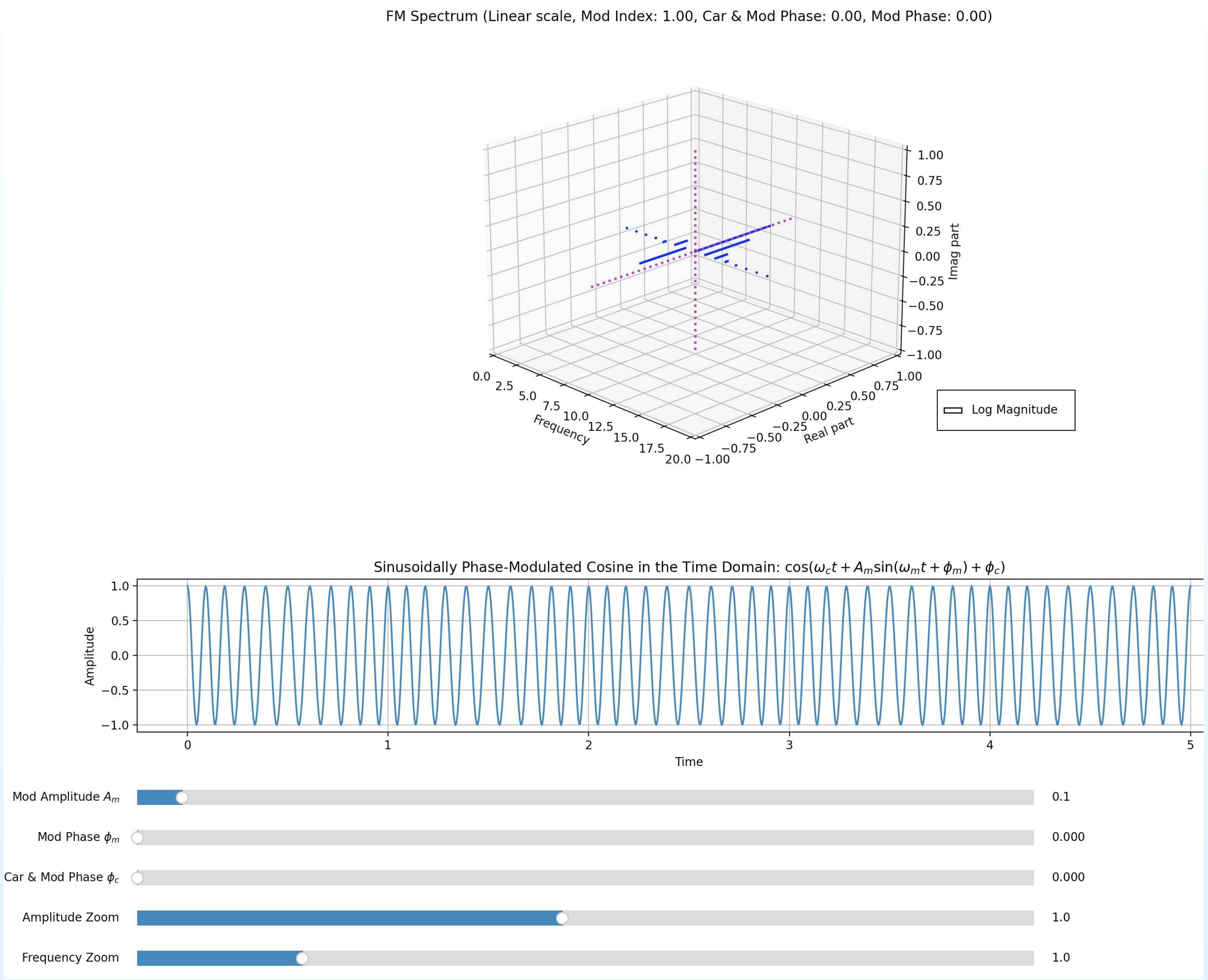
FM Spectra Visualized

To visualize FM spectra interactively, the following prompt did well with Claude Sonnet 3.5:

> Write an interactive Python program that displays FM spectra, where the sinusoidal carrier amplitude is 1, the carrier phase is 0, and both the amplitude and phase of the modulating sinusoid are brought out on sliders.

Additional prompts yielded an interactive 3D version:
`http://ccrma.stanford.edu/~jos/python/fm_spectra_3d.py`

fm_spectra_3d.py



Mod Amplitude A_m 0.1

Mod Phase ϕ_m 0.000

Car & Mod Phase ϕ_c 0.000

Amplitude Zoom 1.0

Frequency Zoom 1.0

Sinusoidal Frequency Modulation

For *sinusoidal* frequency modulation, we can repeat our derivation using $\phi_m = \pm\pi/2$, *i.e.*,

$$u = e^{j(\omega_m t \pm \pi/2)} = \pm j e^{j\omega_m t}.$$

. . .

At this point, I pasted this LaTeX document into Claude 3.5 and asked it to finish the derivation for the '+' case $u = je^{j\omega_m t}$ ($\phi_m = \pi/2$) and $\phi_c = 0$. The results were pretty good, but it could not complete the derivation given the prompts tried.

Below, any edits from me are in [square brackets]. Everything else is Claude 3.5 Sonnet.

Sinusoidal Frequency Modulation

[For *sinusoidal* frequency modulation, we can repeat our derivation using $u = je^{j\omega_m t}$.]
Starting from the Bessel function definition:

$$e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)} = \sum_{n=-\infty}^{\infty} u^n J_n(z)$$

Substituting $u = je^{j\omega_m t}$ and $z = \beta$, we get:

$$e^{\frac{1}{2}\beta\left(je^{j\omega_m t}-\frac{1}{je^{j\omega_m t}}\right)} = \sum_{n=-\infty}^{\infty} \left(je^{j\omega_m t}\right)^n J_n(\beta)$$

Simplifying the left-hand side:

$$e^{j\beta\cos(\omega_m t)} = \sum_{n=-\infty}^{\infty} j^n e^{jn\omega_m t} J_n(\beta)$$

Sinusoidal Frequency Modulation (Continued)

Multiplying both sides by $e^{j\omega_c t}$:

$$e^{j[\omega_c t+\beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} j^n e^{j(\omega_c+n\omega_m)t} J_n(\beta)$$

Using Euler's formula and equating real and imaginary parts [INCORRECT]:

$$\cos[\omega_c t + \beta\cos(\omega_m t)] = \sum_{n=-\infty}^{\infty} (-1)^n J_{2n}(\beta)\cos[(\omega_c + 2n\omega_m)t]$$

$$\sin[\omega_c t + \beta\cos(\omega_m t)] = \sum_{n=-\infty}^{\infty} (-1)^n J_{2n+1}(\beta)\sin[(\omega_c + (2n+1)\omega_m)t]$$

This represents [co]sinusoidal *phase modulation* by $\Theta(t) = \beta\cos(\omega_m t)$.
[The righthand sides of the real and imaginary parts are incorrect. We'll fix this later.]

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Analysis of Sinusoidal Frequency Modulation

The instantaneous phase for this modulation is:

$$\Theta(t) = \omega_c t + \beta \cos(\omega_m t)$$

The instantaneous frequency is therefore:

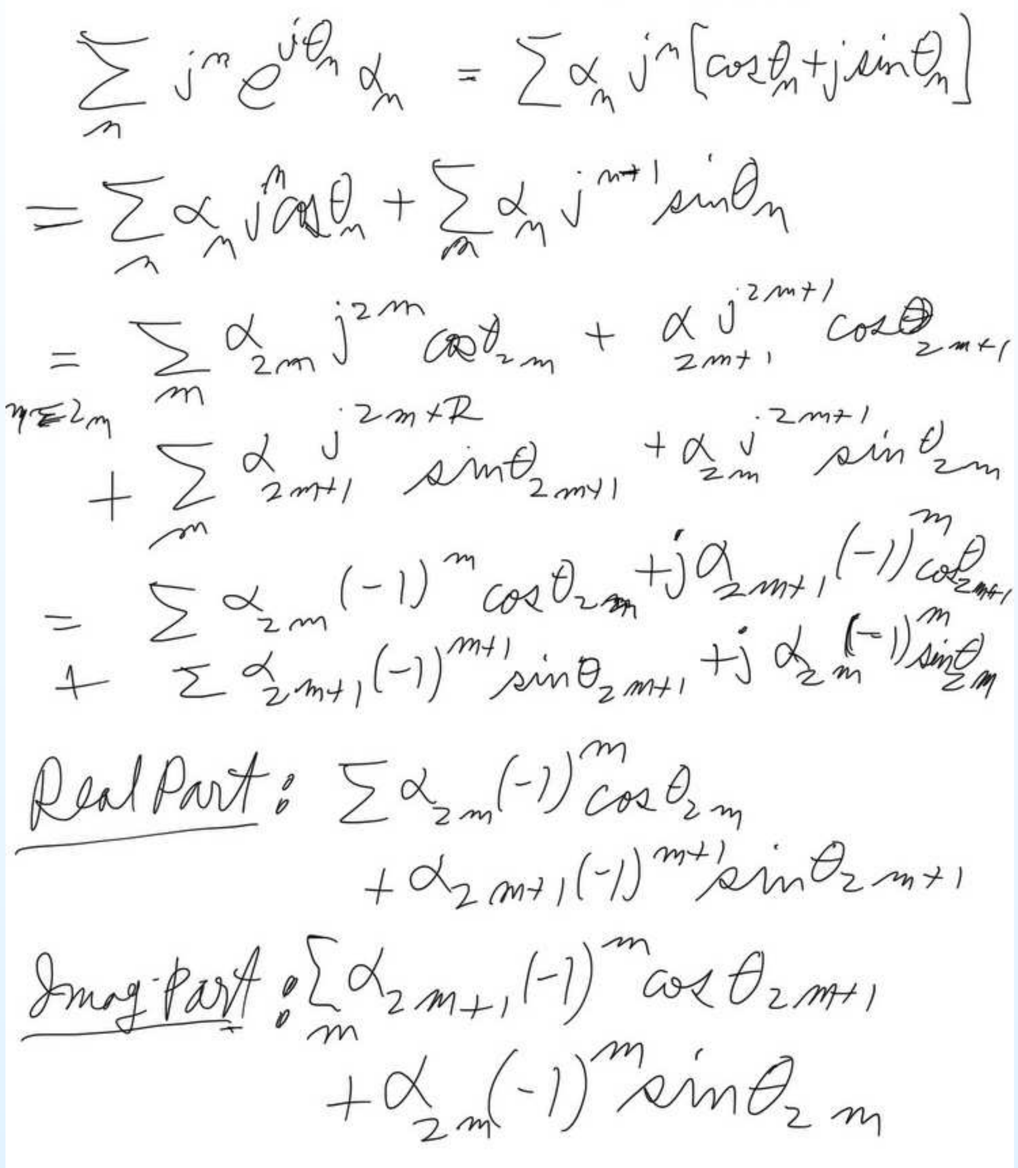
$$\dot{\Theta}(t) = \omega_c - \beta \omega_m \sin(\omega_m t)$$

We can observe that:

- The carrier frequency is ω_c
- The modulation index is β
- The modulation frequency is ω_m
- The peak frequency deviation is $\beta\omega_m$

This completes our derivation of sinusoidal frequency modulation, showing its relationship to Bessel functions and its spectral characteristics.

Hand derivation was input to Claude as a copy/pasted Notes screenshot



$$\sum_n j^n e^{j\theta_n} \alpha_n = \sum \alpha_n j^n [\cos\theta_n + j\sin\theta_n]$$

$$= \sum_n \alpha_n j^n \cos\theta_n + \sum_n \alpha_n j^{n+1} \sin\theta_n$$

$n = 2m$

$$= \sum_m \alpha_{2m} j^{2m} \cos\theta_{2m} + \alpha_{2m+1} j^{2m+1} \cos\theta_{2m+1}$$

$$+ \sum_m \alpha_{2m+1} j^{2m+2} \sin\theta_{2m+1} + \alpha_{2m} j^{2m+1} \sin\theta_{2m}$$

$$= \sum \alpha_{2m} (-1)^m \cos\theta_{2m} + j\alpha_{2m+1} (-1)^m \cos\theta_{2m+1}$$

$$+ \sum \alpha_{2m+1} (-1)^{m+1} \sin\theta_{2m+1} + j\alpha_{2m} (-1)^m \sin\theta_{2m}$$

Real Part: $\sum \alpha_{2m}(-1)^m \cos\theta_{2m}$

$+ \alpha_{2m+1}(-1)^{m+1} \sin\theta_{2m+1}$

Imag Part: $\sum_m \alpha_{2m+1}(-1)^m \cos\theta_{2m+1}$

$+ \alpha_{2m}(-1)^m \sin\theta_{2m}$

Hand Derivation Typeset by Claude from a Screenshot

$$\begin{aligned}
\sum_n j^n e^{j\theta_n}\alpha_n &= \sum_n \alpha_n j^n(\cos\theta_n + j\sin\theta_n) = \sum_n \alpha_n j^n \cos\theta_n + \sum_n \alpha_n j^{n+1}\sin\theta_n \\
&= \sum_m \alpha_{2m} j^{2m}\cos\theta_{2m} + \alpha_{2m+1} j^{2m+1}\cos\theta_{2m+1} \\
&\quad + \sum_m \alpha_{2m+1} j^{2m+2}\sin\theta_{2m+1} + \alpha_{2m} j^{2m+1}\sin\theta_{2m} \\
&= \sum_m \alpha_{2m}(-1)^m\cos\theta_{2m} + j\alpha_{2m+1}(-1)^m\cos\theta_{2m+1} \\
&\quad + \sum_m \alpha_{2m+1}(-1)^{m+1}\sin\theta_{2m+1} + j\alpha_{2m}(-1)^m\sin\theta_{2m}
\end{aligned}$$

Real Part: $\displaystyle\sum_m \alpha_{2m}(-1)^m\cos\theta_{2m} + \alpha_{2m+1}(-1)^{m+1}\sin\theta_{2m+1}$

Imag Part: $\displaystyle\sum_m \alpha_{2m+1}(-1)^m\cos\theta_{2m+1} + \alpha_{2m}(-1)^m\sin\theta_{2m}$

Online Resources

- These Overheads: `http://ccrma.stanford.edu/~jos/pdf/FMSpectra-2024-08-22.pdf`
  Video Presentation (v1):
  `https://ccrma.stanford.edu/~jos/mp4/FMSpectra-2024-08-22-E1-C.mp4`
- Python Code for Visualizing FM Spectra:
  `http://ccrma.stanford.edu/~jos/python/fm_spectra_3d.py`
- Proof of Euler's Identity (Music 320):
  `https://ccrma.stanford.edu/~jos/mdft/Proof_Euler_s_Identity.html`
- Bill Schottstaedt's FM Intro: `https://ccrma.stanford.edu/software/snd/snd/fm.html`
- Plotting FM Spectra in Mathematica, by Craig Sapp (1995):
  - Mathematica package source code:
    `https://ccrma.stanford.edu/software/scmp/FMPlot/FMPlot.m`
  - PDFs of Mathematica notebooks using the source code:
    `https://ccrma.stanford.edu/software/scmp/FMPlot/aboutFM.pdf`
    `https://ccrma.stanford.edu/software/scmp/FMPlot/FMPlot-tutorial.pdf`
- **Treatise on the Theory of Bessel Functions** by G. N. Watson (1944):
  `https://www.forgottenbooks.com/en/download/ATreatiseontheTheoryofBesselFunctions_10019747.pdf`

Interesting but Wrong AI Derivations by Claude 3.5

Positive Sinusoidal Frequency Modulation

[To obtain frequency modulation by $\beta\omega_m \sin(\omega_m t)$ instead of $-\beta\omega_m \sin(\omega_m t)$, my next prompt was: “Please complete the section on ”Sinusoidal Frequency Modulation” using the substitution $u = -je^{j\omega_m t}$ instead of $u = je^{j\omega_m t}$” which gave the following.]

For *[positive] sinusoidal* frequency modulation, we can repeat our derivation using $u = -je^{j\omega_m t}$. Starting from the Bessel function definition:

$$e^{\frac{1}{2}z\left(u-\frac{1}{u}\right)} = \sum_{n=-\infty}^{\infty} u^n J_n(z)$$

Substituting $u = -je^{j\omega_m t}$ and $z = \beta$, we get:

$$e^{\frac{1}{2}\beta\left(-je^{j\omega_m t}-\frac{1}{-je^{j\omega_m t}}\right)} = \sum_{n=-\infty}^{\infty} \left(-je^{j\omega_m t}\right)^n J_n(\beta)$$

Simplifying the left-hand side:

$$e^{-j\beta\cos(\omega_m t)} = \sum_{n=-\infty}^{\infty} (-j)^n e^{jn\omega_m t} J_n(\beta)$$

Positive Sinusoidal Frequency Modulation (Continued)

Multiplying both sides by $e^{j\omega_c t}$:

$$e^{j[\omega_c t-\beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n e^{j(\omega_c+n\omega_m)t} J_n(\beta)$$

[This much looks correct, and it shows that all we have to do is replace β by $-\beta$ in our previous derivation of negative sinusoidal frequency modulation. On the right-hand side this changes $J_n(\beta)$ to $J_n(-\beta)=(-1)^n J_n(\beta)$ (you can see this in our starting definition).]

Using Euler's formula and equating real and imaginary parts [INCORRECT]:

$$\cos[\omega_c t-\beta\cos(\omega_m t)] = \sum_{n=-\infty}^{\infty} J_{2n}(\beta)\cos[(\omega_c+2n\omega_m)t]$$

$$\sin[\omega_c t-\beta\cos(\omega_m t)] = \sum_{n=-\infty}^{\infty} -J_{2n+1}(\beta)\sin[(\omega_c+(2n+1)\omega_m)t]$$

This represents sinusoidal *phase modulation* by $\Theta(t)=-\beta\cos(\omega_m t)$.

Positive Sinusoidal Frequency Modulation, Concluded

The instantaneous phase for this modulation is:

$$\Theta(t) = \omega_c t - \beta\cos(\omega_m t)$$

The instantaneous frequency is therefore:

$$\dot{\Theta}(t) = \omega_c + \beta\omega_m \sin(\omega_m t)$$

We can observe that:

- The carrier frequency is ω_c
- The modulation index is β
- The modulation frequency is ω_m
- The peak frequency deviation is $\beta\omega_m$

This completes our derivation of sinusoidal frequency modulation, showing its relationship to Bessel functions and its spectral characteristics. [But it was *not* derived, and it guessed wrong sometimes.]

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Claude Flails
- +Sine FM
- Claude-2b
- Claude-3b
- Claude-2c
- Claude-2d1
- Claude-2d2
- Claude-2e1
- Claude-2e2

ChatGPT and Gemini

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Another Claude attempt, trying to take smaller steps

Multiplying both sides by $e^{j\omega_c t}$:

$$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n e^{j(\omega_c + n\omega_m)t} J_n(\beta)$$

Using Euler's formula and equating real and imaginary parts: [*Bzzzt*]

$$\cos[\omega_c t - \beta\cos(\omega_m t)] = J_0(\beta)\cos(\omega_c t) + \sum_{n=1}^{\infty} J_{2n}(\beta)[\cos((\omega_c + 2n\omega_m)t) + \cos((\omega_c - 2n\omega_m)t)]$$

$$\sin[\omega_c t - \beta\cos(\omega_m t)] = 2\sum_{n=1}^{\infty} J_{2n-1}(\beta)[\sin((\omega_c + (2n-1)\omega_m)t) - \sin((\omega_c - (2n-1)\omega$$

This represents sinusoidal *phase modulation* by $\Theta(t) = -\beta\cos(\omega_m t)$.

Sinusoidal Frequency Modulation (Continued)

[Prompt ≈ Apply Euler's identity on the righthand side first . . .]

Starting with:

$$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n e^{j(\omega_c + n\omega_m)t} J_n(\beta)$$

Applying Euler's identity to the right-hand side:

$$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n [\cos((\omega_c + n\omega_m)t) + j\sin((\omega_c + n\omega_m)t)] J_n(\beta)$$

$$= \sum_{n=-\infty}^{\infty} [(-j)^n \cos((\omega_c + n\omega_m)t) + (-j)^{n+1} \sin((\omega_c + n\omega_m)t)] J_n(\beta)$$

[Looks promising!]

Sinusoidal Frequency Modulation (Continued)

Now, let's separate the even and odd terms of n:

$$\begin{aligned} e^{j[\omega_c t - \beta\cos(\omega_m t)]} &= \sum_{k=-\infty}^{\infty} [(1)^k \cos((\omega_c + 2k\omega_m)t) + (-j)(1)^k \sin((\omega_c + 2k\omega_m)t)] J_{2k}(\beta) \\ &+ \sum_{k=-\infty}^{\infty} [(-j)(1)^k \cos((\omega_c + (2k+1)\omega_m)t) + (-1)^{k+1} \sin((\omega_c + (2k+1)\omega_m)t)] J_{2k+1}(\beta) \end{aligned}$$

Simplifying:

$$\begin{aligned} e^{j[\omega_c t - \beta\cos(\omega_m t)]} &= \sum_{k=-\infty}^{\infty} [\cos((\omega_c + 2k\omega_m)t) - j\sin((\omega_c + 2k\omega_m)t)] J_{2k}(\beta) \\ &+ \sum_{k=-\infty}^{\infty} [-j\cos((\omega_c + (2k+1)\omega_m)t) - \sin((\omega_c + (2k+1)\omega_m)t)] J_{2k+1}(\beta) \end{aligned}$$

[No cigar]

Sinusoidal Frequency Modulation (Continued)

[Final Prompt: The real part will contain all the cosine terms from the even indices and all the sine terms from the odd indices. And vice versa for the imaginary part.]

Starting with:

$$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n e^{j(\omega_c + n\omega_m)t} J_n(\beta)$$

Applying Euler's identity and reorganizing terms:

$$\begin{aligned}
e^{j[\omega_c t - \beta\cos(\omega_m t)]} &= \sum_{n=-\infty}^{\infty} (-j)^n [\cos((\omega_c + n\omega_m)t) + j\sin((\omega_c + n\omega_m)t)] J_n(\beta) \\
&= \left[\sum_{k=-\infty}^{\infty} \cos((\omega_c + 2k\omega_m)t) J_{2k}(\beta)\right] + \left[\sum_{k=-\infty}^{\infty} \sin((\omega_c + (2k+1)\omega_m)t) J_{2k+1}(\beta)\right] \\
&+ j\left[\sum_{k=-\infty}^{\infty} \sin((\omega_c + 2k\omega_m)t) J_{2k}(\beta)\right] - j\left[\sum_{k=-\infty}^{\infty} \cos((\omega_c + (2k+1)\omega_m)t) J_{2k+1}(\beta)\right]
\end{aligned}$$

Sinusoidal Frequency Modulation (Continued)

Therefore, we can write:

$$\begin{aligned}\cos[\omega_c t - \beta\cos(\omega_m t)] = &\sum_{k=-\infty}^{\infty} \cos((\omega_c + 2k\omega_m)t) J_{2k}(\beta)\\ &+ \sum_{k=-\infty}^{\infty} \sin((\omega_c + (2k+1)\omega_m)t) J_{2k+1}(\beta)\end{aligned}$$

$$\begin{aligned}\sin[\omega_c t - \beta\cos(\omega_m t)] = &\sum_{k=-\infty}^{\infty} \sin((\omega_c + 2k\omega_m)t) J_{2k}(\beta)\\ &- \sum_{k=-\infty}^{\infty} \cos((\omega_c + (2k+1)\omega_m)t) J_{2k+1}(\beta)\end{aligned}$$

[This looks better, but the alternating minus signs were lost along the way. I think maxima or Mathematica should be tried next, or just do a good old derivation by hand! [which I later did]]

ChatGPT-4o and Gemini 2024-07-25 Flail

Sinusoidal Frequency Modulation

[I tried ChatGPT-4o as well, and here it is.]

For *sinusoidal* frequency modulation, we can repeat our derivation using $u = je^{j\omega_m t}$. Starting from the expression for phase modulation:

$$e^{j[\omega_c t+\beta\sin(\omega_m t)]} = \sum_{n=-\infty}^{\infty} e^{j(\omega_c+n\omega_m)t} J_n(\beta),$$

we know that the instantaneous phase is given by

$$\Theta(t) = \omega_c t + \beta\sin(\omega_m t).$$

Sinusoidal Phase Modulation, Continued, ChatGPT

[Unlike Claude 3.5, ChatGPT-4o tried to jam everything onto one slide, when at least three were required. I introduced all slide-breaks manually myself.]

For sinusoidal frequency modulation, the instantaneous frequency is defined as

$$\omega_i(t) = \omega_c + \Delta\omega \sin(\omega_m t),$$

where $\Delta\omega$ is the peak frequency deviation. Integrating the instantaneous frequency to obtain the phase, we have

$$\Theta(t) = \omega_c t - \frac{\Delta\omega}{\omega_m} \cos(\omega_m t).$$

[I like this introduction of desired instantaneous frequency that is integrated to obtain desired instantaneous phase.]

Sinusoidal Frequency Modulation

We define the modulation index β as

$$\beta = \frac{\Delta\omega}{\omega_m},$$

so that

$$\Theta(t) = \omega_c t - \beta\cos(\omega_m t).$$

Rewriting the expression in terms of β, we get

$$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} e^{j(\omega_c + n\omega_m)t} J_n(\beta).$$

[This formula has not been derived. Error!]

Sinusoidal Frequency Modulation

By using the property of Bessel functions that $J_{-n}(z) = (-1)^n J_n(z)$, we can rewrite the above as

$$e^{j[\omega_c t + \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} e^{j(\omega_c + n\omega_m)t} J_n(\beta).$$

This expression represents the spectrum of the sinusoidally frequency-modulated signal, showing that it consists of a carrier at frequency ω_c and sidebands spaced at multiples of the modulation frequency ω_m, with amplitudes given by the Bessel functions $J_n(\beta)$. The complete signal can be expressed as

$$\cos\left[\omega_c t + \beta\sin(\omega_m t)\right] = \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[(\omega_c + n\omega_m)t].$$

[How are we suddenly back to sinusoidal modulation? Poor! Wrap-up summary paragraph deleted.]

- Frequency Modulation (FM)
- Watson
- Bessel Coefficients
- Specialization to FM/PM
- Phase Modulation
- Frequency Modulation
- FM Visualized
- 3D Spectral Plot
- Sinusoidal FM
- Claude-1
- Claude-2
- Claude-3
- Claude Input
- Claude Corrected
- Online Resources

Claude Flails

ChatGPT and Gemini
- ChatGPT-1
- ChatGPT-2
- ChatGPT-3
- ChatGPT-4
- Gemini

All Three

Sinusoidal Frequency Modulation (cont’d)

[Here is Gemini 2024.07.25. While ChatGPT’s inference seemed instant, Gemini’s rolled out more slowly.]

Setting $u = je^{j\omega_m t}$ in the Bessel generating function gives

$$\begin{aligned} e^{\frac{1}{2}\beta\left(je^{j\omega_m t}-\frac{1}{je^{j\omega_m t}}\right)} &= \sum n=-\infty^{\infty}(je^{j\omega_m t})^n J_n(\beta) \\ e^{j\beta\cos(\omega_m t)} &= \sum n=-\infty^{\infty} j^n e^{jn\omega_m t} J_n(\beta). \end{aligned}$$

Multiplying both sides by $e^{j\omega_c t}$ and separating real and imaginary parts yields

$$\begin{aligned} \cos\left[\omega_c t+\beta\cos(\omega_m t)\right] &= \sum n=-\infty^{\infty} J_n(\beta)\cos\left[(\omega_c+n\omega_m)t+\frac{n\pi}{2}\right], \\ \sin\left[\omega_c t+\beta\cos(\omega_m t)\right] &= \sum n=-\infty^{\infty} J_n(\beta)\sin\left[(\omega_c+n\omega_m)t+\frac{n\pi}{2}\right]. \end{aligned}$$

[Subsequent nonsense deleted. Only Gemini’s text did not even compile (ω_m became `\omegam` which was undefined).]

All Three on their Shared Pitfall

Claude

Prompt: In LaTeX, find the real and imaginary parts of
$e^{j[\omega_c t - \beta\cos(\omega_m t)]} = \sum_{n=-\infty}^{\infty} (-j)^n e^{j(\omega_c + n\omega_m)t} J_n(\beta)$
Claude, syntax-corrected by ChatGPT (didn't compile at first)

$$\begin{aligned}
\text{Re}\{e^{j[\omega_c t - \beta\cos(\omega_m t)]}\} &= \cos[\omega_c t - \beta\cos(\omega_m t)] \\
&= \sum_{n \text{ even}} (-1)^{n/2} \cos[(\omega_c + n\omega_m)t] J_n(\beta) \\
&\quad + \sum_{n \text{ odd}} (-1)^{(n-1)/2} \sin[(\omega_c + n\omega_m)t] J_n(\beta)
\end{aligned}$$

$$\begin{aligned}
\text{Im}\{e^{j[\omega_c t - \beta\cos(\omega_m t)]}\} &= \sin[\omega_c t - \beta\cos(\omega_m t)] \\
&= \sum_{n \text{ even}} (-1)^{n/2} \sin[(\omega_c + n\omega_m)t] J_n(\beta) \\
&\quad - \sum_{n \text{ odd}} (-1)^{(n-1)/2} \cos[(\omega_c + n\omega_m)t] J_n(\beta)
\end{aligned}$$

ChatGPT

Real Part:

$$\text{Re}\left\{\sum_{n=-\infty}^{\infty}(-j)^n e^{j(\omega_c+n\omega_m)t}J_n(\beta)\right\} = \sum_{n=-\infty}^{\infty}\left[\cos(n\pi/2)\cos((\omega_c+n\omega_m)t)+\sin(n\pi/2)\sin((\omega_c+n\omega_m)t)\right]$$

Imaginary Part:

$$\text{Im}\left\{\sum_{n=-\infty}^{\infty}(-j)^n e^{j(\omega_c+n\omega_m)t}J_n(\beta)\right\} = \sum_{n=-\infty}^{\infty}\left[\cos(n\pi/2)\sin((\omega_c+n\omega_m)t)-\sin(n\pi/2)\cos((\omega_c+n\omega_m)t)\right]$$

Gemini

$$\begin{aligned} e^{j[\omega_c t-\beta\cos(\omega_m t)]} &= \cos(\omega_c t-\beta\cos(\omega_m t))+j\sin(\omega_c t-\beta\cos(\omega_m t)) \\ &= \cos(\omega_c t)\cos(\beta\cos(\omega_m t))+\sin(\omega_c t)\sin(\beta\cos(\omega_m t)) \\ &\quad + j[\sin(\omega_c t)\cos(\beta\cos(\omega_m t))-\cos(\omega_c t)\sin(\beta\cos(\omega_m t))] \end{aligned}$$

$$\boxed{a=\cos(\omega_c t)\cos(\beta\cos(\omega_m t))+\sin(\omega_c t)\sin(\beta\cos(\omega_m t))}$$

$$\boxed{b=\sin(\omega_c t)\cos(\beta\cos(\omega_m t))-\cos(\omega_c t)\sin(\beta\cos(\omega_m t))}$$

Fairness of Comparisons

The above makes Claude 3.5 look a *lot* better than ChatGPT-4o and Gemini 2024-07-25. However, this comparison is unfair because I used only Claude to typeset the initial formulas in LaTeX (from a copy/pasted screenshot from Watson and formula substition directions—see below for a link), and that chat was still in its context buffer. ChatGPT and Gemini only got one prompt to apply Euler to the final correct identity.

I do find Claude 3.5 to be the best overall, but its margin is slim over ChagGPT, and Gemini won one competition. Also, I've done only a handful of head-to-head comparisons so far, less than 10 for sure.

Laurent Series

Suppose we can expand $f(z)$ in a Laurent series about $z=0$:

$$f(z) = \sum_{m=-\infty}^{\infty} a_m\, z^m \;=\; \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \cdots$$

Our task is to find a_n for $n \in \mathbb{Z}$. Assume $f(z)$ is *analytic* (differentiable of all orders) in $0 < |z| \le r$. Then the integral around the circle of radius r centered on $z=0$ gives

$$\begin{aligned} \int_0^{2\pi} f(re^{i\theta})d\theta &= \cdots + a_{-2}r^{-2}\int_0^{2\pi} e^{-2i\theta}d\theta + a_{-1}r^{-1}\int_0^{2\pi} e^{-i\theta}d\theta \\ +\; a_0\int_0^{2\pi} d\theta &+ a_1\, r\int_0^{2\pi} e^{i\theta}d\theta + a_2\, r^2\int_0^{2\pi} e^{2i\theta}d\theta + \cdots \\ &= a_0 \cdot 2\pi \end{aligned}$$

Thus, $\boxed{a_n = \frac{1}{2\pi}\int_0^{2\pi} z^{-n} f(z)\Big|_{z=e^{i\theta}} d\theta}$

Contour Integral Formulation

We have

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} z^{-n} f(z)\Big|_{z=e^{i\theta}} d\theta.$$

For $z = r\,e^{i\theta}$ and constant r, we can write

$$dz = i\,r\,e^{i\theta}d\theta = i\,z\,d\theta$$

or

$$d\theta = \frac{1}{i}\frac{dz}{z}$$

so that

$$a_n = \frac{1}{2\pi i}\oint z^{-n} f(z)\frac{dz}{z}$$

where the contour of integration can be the zero-centered circle of radius r, or any other analytic path through the annulus on which $f(z)$ is analytic.