Final Exam Practice Questions for Music 421, with Solutions

Elementary Fourier Relationships

1. For the window $w = [1/\sqrt{2}, 1, 1/\sqrt{2}]$, what is

   (a) the dc magnitude of the window transform?

   Solution: $\boxed{1+\sqrt{2}}$

   (b) the magnitude at half the sampling rate?

   Solution: $\boxed{1-\sqrt{2}}$

2. Consider a discrete-time chirp signal $x(nT) \stackrel{\Delta}{=} \cos\left[\alpha \cdot (nT)^2 + \phi_o\right]$, where $\alpha = \frac{\pi}{10}\text{rad}/s^2$, $T = \frac{1}{f_s} = \frac{1}{44{,}100}$ sec. How long does it take for the instantaneous frequency to sweep from DC to the Nyquist limit at 22,050Hz ?

   Solution: $\boxed{5 \cdot 44100 \text{ seconds}}$

3. For the frequency response

$$H(\omega) = \begin{cases} 1, & 0 \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the corresponding *continuous-time* impulse response $h(t)$.
   Solution:

$$\boxed{h(t) = \frac{1}{2} e^{j\frac{\pi}{2}t} \text{sinc}\left(\frac{t}{2}\right)}$$

   (b) Find $h(n)$ for $n = 0, \pm 1, \pm 2, \ldots$. ($T = 1$.) Simplify the result as far as you can.
   Solution:

$$\boxed{h(n) = \frac{1}{2} e^{j\frac{\pi}{2}n} \text{sinc}\left(\frac{n}{2}\right) = \cdots}$$

   (c) Find $h(n+1/2)$ for $n = 0, \pm 1, \pm 2, \ldots$.
   Solution:

$$\boxed{h(n+1/2) = \frac{1}{2} e^{j\frac{\pi}{2}(n+\frac{1}{2})} \text{sinc}\left(\frac{n+\frac{1}{2}}{2}\right) = \cdots}$$

Window Method for FIR Filter Design

1. Describe, in a step-by-step manner, the window method for FIR design.

   Solution: See text.

2. If the window $w(n) = \text{sinc}(n/10)$, viewed as the impulse response of a lowpass filter, is said to "cut off" at $\omega = \pi/10$, at what frequency does the window $w^4(n)$ cut off?

   Solution: Squaring the window corresponds to convolving the window transform with itself, thus moving its cut-off from $\pi/10$ to $2\pi/10$. Repeating this operation corresponds to $w^4(n)$ and moves the cut-off from $2\pi/10$ to $\boxed{2\pi/5}$. The pass-band shape goes from rectangular to triangular after one convolution, and then to quadratic. Thus, it's not really correct to call it a "lowpass" filter in the usual sense.

3. If $h(n)$ is the impulse response of a lowpass filter, describe two ways to make a highpass filter from it.

   Solution: See text. The best method covered is a π-rotation of the spectrum, $\boxed{(-1)^n h(n)}$. A second method mentioned is forming $1 - H(z) \leftrightarrow \delta(n) - h(n)$, but this requires more careful normalization of h.

4. We wish to design an ideal lowpass filter $H(z)$ with cut-off frequency at $f_s/4 = 0.25$ in normalized frequency. Suppose we want to compute the ideal sampled impulse response by simulating the IDTFT using a large IFFT. Find the closed-form expression for the ideal impulse response $h(n)$ and use it to determine the smallest FFT length N (a power of 2) which guarantees that every time-aliased sample of the inverse DTFT will have amplitude less than $1/2000\pi$.

   Solution: The sinc fn reaches $1/2000\pi$ at $n = 1000$, so $N = 2048$ is first FFT size within specs.

5. A causal Hann window w(n) can be implemented in the frequency domain as the smoothing kernel $W = (-1/4, 1/2, -1/4)$. We know that windowing in the time domain corresponds to smoothing in the frequency domain. The second derivative operator ("Laplacian") in the frequency domain is given by the kernel $(1, -2, 1) = -4W$. This kernel is often used as an "edge detector" in image processing, and it is regarded as a kind of high-pass filter (a double differentiation). Explain how a Hann window in the time domain can be considered both a smoothing operation and an edge detector (highpass filter) in the frequency domain.

   Solution: One is the π-rotation of the other.

FFT Convolution

1. Suppose we perform FFT based convolution using the following system parameters:

$$
\begin{aligned}
N &= 8 \quad \text{(FFT length)} \\
M &= 5 \quad \text{(Window length)} \\
x(n) &= (-1)^n u(n) \quad \text{(Input signal)} \\
w(n) &= \text{Triangular window} = [1, 2, 3, 2, 1]
\end{aligned}
$$

(a) What is the FFT input buffer for frame 0, assuming

i. Causal processing

Solution: $\boxed{[1, -2, 3, -2, 1, 0, 0, 0]}$

ii. Zero-phase processing

Solution: $\boxed{[3, -2, 1, 0, 0, 0, 0, 0]}$

(b) What is the FFT input buffer for frame 1, assuming a hop size $R = 2$, and

i. Causal processing

Solution: $\boxed{[1, -2, 3, -2, 1, 0, 0, 0]}$

ii. Zero-phase processing

Solution: $\boxed{[3, -2, 1, 0, 0, 0, 1, -2]}$

(c) What is the overlap-add of the window (for all n), given the hop size $R = 2$?

Solution: $\boxed{\text{ALIAS}_2(w) = [5, 4]}$

(d) What is the overlap-add of the window (for all n), given the hop size $R = 1$?

Solution: $\boxed{\text{ALIAS}_1(w) = \sum_n w(n) = 9}$

(e) What is the frame rate in Hz when $R = 2$ and the sampling rate is 60Hz?

Solution: $\boxed{30 \text{ Hz}}$

(f) Find the closed form expression of $W(\omega)$ in the zero-phase case (using the bilateral DTFT of w)?

Solution: $\boxed{3 + 4\cos(\omega T) + 2\cos(2\omega T)}$

(g) Plot the magnitude spectrum $|W(\omega)|$ on a linear scale.

Solution: Paste the following into a matlab command window:

```
wT = [-pi:0.1:pi];
W = 3*cos(0*wT) + 4*cos(wT) + 2*cos(2*wT);
plot(wT,abs(W)); grid on;
```

Find

i. $W(0)$

Solution: $\boxed{W(0) = 3 + 4 + 2 = 9}$

ii. $W(\pi/T)$

Solution: $\boxed{W(\pi) = 3 - 4 + 2 = 1}$

(h) Find

$$\frac{1}{R}\sum_k W(\omega_k)e^{j\omega_k nT}$$

where $\omega_k \triangleq 2\pi k f_s/R$, and $R = 2$.

Solution:

$$\begin{aligned}\frac{1}{2}\sum_{0}^{1} W(\omega_k)e^{j\omega_k nT} &= \frac{1}{2}\left[W(0)+W(1)e^{j\pi n}\right] \\ &= \frac{9}{2}+(-1)^n\frac{1}{2} = 4.5+(-1)^n 0.5 \\ &= \begin{cases} 5, & n \text{ even} \\ 4, & n \text{ odd}\end{cases}\end{aligned}$$

or

$$\boxed{\text{REPEAT}_{\infty}([5,4])}$$

(i) Explain the relationship between the previous answer and that in part 1c.

Solution: By the Poisson Summation Formula, they $\boxed{\text{must be the same.}}$

2. Recall that FFT windows used for long convolutions must satisfy the constant overlap-add (COLA) constraint:

$$\sum_m w(n-mR) = 1$$

Recall also that if this constraint is not satisfied, the window overlap add becomes some R-periodic signal

$$s_R(n) \stackrel{\Delta}{=} \sum_m w(n-mR),$$

and $s_R(n+kR) = s_R(n)$ for every integer k. If $s_R(n)$ is not a constant in this way, can we simply divide by it? If not, show what goes wrong. If so, prove the same result is obtained as if $s_R(n)$ is 1. Consider three cases:

(a) No modifications, i.e., $Y_m(\omega_k) = X_m(\omega_k)$, where m is the frame index, X_m is the FFT of the mth windowed input data frame, and Y_m is the mth spectral frame for the output.

Solution: $\boxed{\text{ok, but not robust in the presence of noise.}}$

(b) Linear time-invariant modifications $[Y_m(\omega_k) = H(\omega_k)X_m(\omega_k)]$.

Solution: $\boxed{\text{Fails}}$ because amplitude modulation is applied to the frame x_m which is then *convolved* with the LTI modification h.

(c) Linear time-varying modifications $[Y_m(\omega_k) = H_m(\omega_k)X_m(\omega_k)]$.

Solution: $\boxed{\text{Fails}}$ for the same reason as the LTI case.

Spectral Modeling

1. Given the spectral magnitude samples $X(\omega_k) = [0, 2, 2, 1, 0, \ldots]$, a local magnitude peak is found to exist in bins $k = 1, 2, 3$, using quadratic interpolation, find

(a) the index k^* of the interpolated peak (in bins),

Solution: $\boxed{1\frac{1}{2}}$

(b) the magnitude of the interpolated peak,

Solution: $\boxed{2\frac{1}{8}}$

(c) the formula for the parabolic interpolation itself.

Solution: See $\boxed{\text{p. 40}}$ of the text.

(d) Explain how to compute the *phase* of the interpolated peak $X(\omega_{k^*})$, and give a formula for it.

Solution: See text — $\boxed{\text{linear interpolation}}$ often good enough.

2. How does the phase vocoder relate to sinusoidal modeling?

   Solution: The instantaneous amplitude and frequency is measured at the output of each filter channel. These functions of time are used to define the amplitude- and frequency-envelopes of quasi-sinusoidal oscillators in a sinusoidal signal model (additive synthesis).

3. Consider two sinusoids at 51 Hz and 200 Hz respectively. The sampling rate is 1500 Hz. Which of the following windows will allow you to resolve the two sinusoids? (Use the definition of "resolve" as in homework 3, where adjacent mainlobes do not overlap.)

Window type	Window length (M)	Able to resolve sinusoids?
Rectangular	100 samples	YES / NO
Hamming	400 samples	YES / NO
Hanning	800 samples	YES / NO
Blackman	900 samples	YES / NO

   Solution: The difference frequency is 149 Hz. We need at least two cycles of the difference frequency under a rectangular window, which, at a 1500 Hz samping rate, is $2 \cdot 1500/149 = 20.134$ samples. Thus, it is easily resolved. The Hamming and Hanning windows require four cycles, or ¿40 samples, so yes, they are both well resolved. The Blackman requires ¿60 samples, so again resolve. The answers are thus $\boxed{\text{YES, YES, YES, YES.}}$ (One suspects there is a typo in this problem statement.)

Filter Banks

1. Given

$$x(n) = \begin{cases} n, & n > 0 \\ 0, & \text{otherwise} \end{cases},$$

   find and plot $y(n)$, where

$$Y(z) \triangleq \frac{1}{2}[X(z) + X(-z)]$$

   Solution:

$$\boxed{y(n) = \begin{cases} n, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}}$$

Relatively hard problems (one or two per final)

1. Consider the basic two-channel filter bank, with lowpass analysis filter $H_0(z)$, highpass analysis filter $H_1(z)$, lowpass synthesis filter $F_0(z)$, and highpass synthesis filter $F_1(z)$:

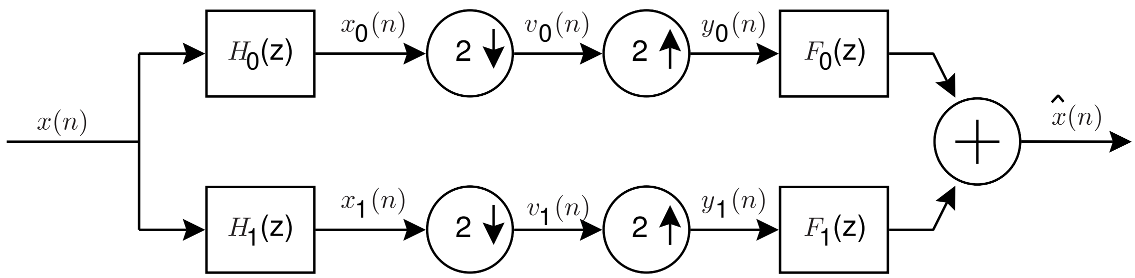

Figure 1: Two-Channel Filter Bank.

(a) Find the input-output transfer function from $x(n)$ to $\hat{x}(n)$.

Solution: See the text, section entitled $\boxed{\text{“Two-Channel Critically Sampled Filter Banks”}}$

(b) Given $H_0(z)$ and $H_1(z)$, find values for $F_0(z)$ and $F_1(z)$ which ensure aliasing cancellation.

(c) Suppose we choose $h_0 = [1, h_0(1), 1, 0, 1, 0, 1, 0]$ and $H_1(z) = 1 - H_0(z)$, with $F_i(z)$ set to cancel aliasing.

Solution: See text, section entitled $\boxed{\text{“Amplitude-Complementary 2-Channel Filter Bank”}}$

i. For what values of $h_0(1)$ is perfect reconstruction obtained?

Solution: $\boxed{\text{Any nonzero value}}$

ii. Find the value $h_0(1)$ which gives unity gain, or prove that unity gain is impossible.

Solution: $\boxed{-\frac{1}{2}}$

iii. Find the delay of the output $\hat{x}(n)$ relative to the input $x(n)$.

Solution: $\boxed{\text{1 sample}}$

(d) Consider using the phase vocoder to analyze a signal which is known to be a sinusoid at some amplitude, frequency, and phase. Suppose that the sinusoid's frequency f_x lies somewhere between channel center frequencies k and $k+1$, i.e., $\omega_k \leq \omega_x \leq \omega_{k+1}$. Find a general formula for combining information from the two channel filters k and $k+1$ in order to produce a more accurate amplitude and frequency estimate (versus time) than would be obtained from using either channel signal alone.

(e) In class, OLA and FBS were derived as alternate interpretations of a single formula, the STFT. In the case of multiplicative spectral modifications (ordinary LTI filtering), we showed that the analysis window $w(n)$ *convolves* with the impulse response of the modification in the OLA case, yielding $w * h$, while it *multiplies* the modifications in the FBS case, yielding $w \cdot h$. Since OLA and FBS are supposed to be two different interpretations of the same STFT formula, explain how the spectral modifications can behave differently in the two cases.

(f) Define $\nu_k(lL) \triangleq X_l(\omega_k)e^{j\omega_k lL}$, where $X_l(\omega_k)$ is the DTFT of the windowed frame x_l sampled at the points $\omega_k = 2\pi k/N$, $x_l(n) \triangleq x(n)w(n-lL)$,and we have $M < N$ for the window length and $L \leq M$ for the frame step-size.

Show that given the window constraint

$$\sum_{l=0}^{L-1} W(\omega_k + l2\pi/L) = Lw(0), \quad k = 0, 1, \ldots, N-1$$

$\nu_k(lL)$ can be up-sampled using ideal (sinc) interpolation to give $\hat{\nu}_k(n)$ for all n, such that

$$\sum_{k=0}^{N-1} \hat{\nu}_k(n) = x(n)$$

Hint: Consider the *dual* of the Poisson summation formula. The usual Poisson summation formula is given by

$$\sum_{l=-\infty}^{\infty} w(n-lL) = \frac{1}{R}\sum_{k=0}^{L-1} W(2\pi k/R)e^{-j2\pi kn/R}$$

(g) We learned in our study of FIR filter design that the length (in seconds) LT required for an FIR lowpass filter impulse response is roughly proportional to one over its transition width B in Hz. Assume for this problem that we have the exact relationship $MT = 2/B$ when the FIR impulse response is designed by the window method using the rectangular window, where T denotes the sampling period.

i. what is the minimum length of an FIR filter which can suppress the negative-frequency component of a single sinusoid at 25 Hz (e.g., for piano analysis) when the sampling rate is $f_s = 10,000$ Hz? For this filter design, the positive-frequency component should lie somewhere in the passband, and the negative-frequency component should lie in somewhere the stopband. State any necessary assumptions you need to arrive at your result.

ii. Is there a name for this type of filter? A few sentences of discussion are invited here.

iii. What is minimum FIR filter length if we change from using a rectangular window to using a Hamming window?

iv. What is minimum FIR filter length if we change from using a rectangular window to using a Blackman window?