MUS420/EE367A Lecture 5C
Choice of Wave Variables in Digital Waveguide Models

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## Outline

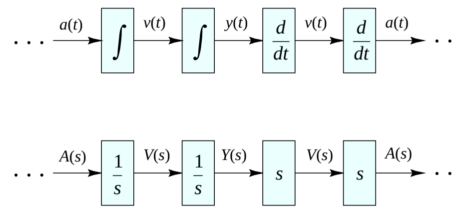
- Displacement, Velocity, Acceleration Waves
- Force Waves
- Root-Power Waves



# Overview of Wave Variable Choices

We have thus far considered only transverse *displacement* waves. We can also choose

- Transverse *velocity* $v \stackrel{\Delta}{=} \dot{y}$
- Transverse *acceleration* $a \stackrel{\Delta}{=} \ddot{y}$
- *Slope* waves $y'$
- *Curvature* waves $y''$ ($= c^2\ddot{y}$ for ideal string)
- Any number of derivatives or integrals of displacement $y$ with respect to time or position
- Conversion between time derivatives carried out by *integrators* and *differentiators*






# Specifying String State

The complete *state* of the string is given at time $n$ by

- $\{y(t_n, x_m), \dot{y}(t_n, x_m)\}_{m=0}^{N-1}$ (typical in acoustics)
- $\{y(t_n, x_m), y(t_{n-1}, x_m)\}_{m=0}^{N-1}$ (typical in acoustic simulations)
- $\{y^+(n-m), y^-(n+m)\}_{m=0}^{N-1}$ (what we did)
- $\{y'^+(n-m), y'^-(n+m)\}_{m=0}^{N-1}$ (today)
- $\{v^+(n-m), v^-(n+m)\}_{m=0}^{N-1}$ (today)
- Any *two* linearly independent variables (either *physical* variables or *wave* variables)
- All traveling-wave variables can be computed from any others, as long as string state is specified
- Wave variable conversions requiring differentiation or integration are relatively expensive since a large-order digital filter is necessary to do it right

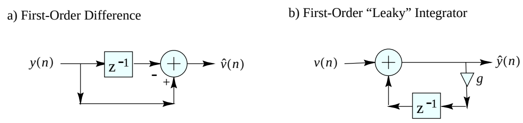


### String State, Cont'd

*Velocity waves* are a good overall choice for strings because

- It is less noisy numerically to integrate for displacement than to differentiate for velocity
- Force (slope) waves = scaling of velocity waves (as we will show shortly)
- Analogous to volume velocity in *acoustic tubes*

## First-Order Discrete-Time Wave-Variable Conversion Filters



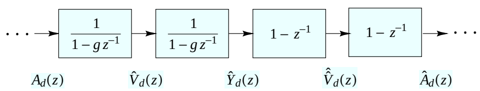

- First-order difference:

$$\hat{v}(n) = y(n) - y(n-1)$$

- First-order "leaky" integrator:

$$\hat{y}(n) = v(n) + g\hat{y}(n-1), \quad g < 1, g \approx 1$$

(loss factor $g$ avoids DC build-up)






## Filter Design Approach

- *Ideal Digital Differentiator:*

$$H(e^{j\omega T}) \approx j\omega, \quad \omega \in [-\pi/T, \pi/T]$$

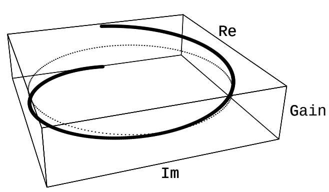
- *Ideal Digital Integrator*

$$H(e^{j\omega T}) \approx \frac{1}{j\omega}, \quad \omega \in [-\pi/T, \pi/T]$$

- Exact match is *not possible* in *finite order*
- Minimize $\left\| H(e^{j\omega T}) - \hat{H}(e^{j\omega T}) \right\|$ where $\hat{H}$ is the digital filter frequency response



## Ideal Differentiator Frequency Response




- Discontinuity at $z = -1$ ensures no exact finite-order solution
- Need *oversampling factor*, as in interpolator design (e.g., $20$ kHz to $22.05$ kHz)
  Response is unconstrained between bandlimit and $f_s/2$
- As before, a small increment in oversampling factor yields a much larger decrease in required filter order to meet a given spec



# Spatial Derivatives

*Slope waves* are simply related to velocity waves.
By the chain rule,

$$\begin{aligned} y'(t,x) &\stackrel{\Delta}{=} \frac{\partial}{\partial x} y(t,x) \\ &= y_r'(t - x/c) + y_l'(t + x/c) \\ &= -\frac{1}{c}\dot{y}_r(t - x/c) + \frac{1}{c}\dot{y}_l(t + x/c) \\ &\rightarrow -\frac{1}{c}v^+(n-m) + \frac{1}{c}v^-(n+m) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} y'^+ &= -\tfrac{1}{c}v^+ \\ y'^- &= \tfrac{1}{c}v^- \end{aligned}$$

or

$$\begin{aligned} v^+ &= -cy'^+ \\ v^- &= cy'^- \end{aligned}$$

- Physical string slope = (lower rail - upper rail)/c in a velocity-wave simulation
- $\Rightarrow v^-(0+m) = v^+(0-m)\,\forall m$ on a struck string

## Wave Impedance

We just showed

$$\begin{aligned} y'^{+} &= -\ \tfrac{1}{c}v^{+} \\ y'^{-} &= \quad \tfrac{1}{c}v^{-} \end{aligned}$$

Define new wave variables in terms of slope waves as

$$\begin{aligned} f^{+} &\stackrel{\Delta}{=} -Ky'^{+} \\ f^{-} &\stackrel{\Delta}{=} -Ky'^{-} \end{aligned}$$

Note that $f^{\pm}$ are in physical units of *force*.
We have

$$\begin{aligned} f^{+} &= \quad \tfrac{K}{c}v^{+} \\ f^{-} &= -\ \tfrac{K}{c}v^{-} \end{aligned}$$

Recall

$$c = \sqrt{\frac{K}{\epsilon}}$$

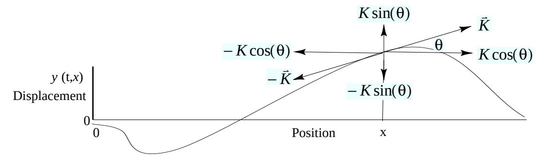
$$\Rightarrow \qquad \frac{K}{c} = \sqrt{K\epsilon} \stackrel{\Delta}{=} R$$

which is the *wave impedance* of the ideal string (force/velocity for traveling waves). Thus,

$$\boxed{\begin{aligned} f^{+} &= \quad R\,v^{+} \\ f^{-} &= -\ R\,v^{-} \end{aligned}}$$



## Force Waves




- Vertical force acting to the left is

$$f_l(t,x) = K\sin(\theta) \approx K\tan(\theta) = Ky'(t,x)$$

- Opposing force, acting to the right, is

$$f_r(t,x) = -K\sin(\theta) \approx -Ky'(t,x)$$

  (Note that a negative slope pulls "up" on the segment to the right)
- These forces must cancel since a nonzero net force on a massless point would produce infinite acceleration.



To unify *vibrating strings* with *acoustic tubes*, we choose the force which *acts to the right* as our *force wave variable*:

$$f(t,x) \stackrel{\Delta}{=} f_r(t,x) = -Ky'(t,x)$$

- Analogous to longitudinal pressure in acoustic tubes
- We have

$$f(t,x) = \frac{K}{c}\left[\dot{y}_r(t-x/c) - \dot{y}_l(t+x/c)\right]$$

- Force waves are thus *proportional* to velocity waves
- Proportionality constant is called the *wave impedance* (or *characteristic impedance*) of the string:

$$R \stackrel{\Delta}{=} \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c$$

- Wave impedance = geometric mean of spring force and inertial mass
- Traveling force-wave components:

$$\begin{aligned} f^{+}(n) &= \quad Rv^{+}(n) \\ f^{-}(n) &= -\ Rv^{-}(n) \end{aligned}$$

- Traveling force is always *in phase* with its velocity (consider left-going minus sign to be associated with direction of travel)



- $f^{+} = Rv^{+}$ is a mechanical counterpart of *Ohm's Law*
- In c.g.s. units, $R$ can be called *acoustical ohms*

For *acoustic tubes*, we have

$$\begin{aligned} p^{+}(n) &= \quad R_{\mathsf{T}}u^{+}(n) \\ p^{-}(n) &= -\ R_{\mathsf{T}}u^{-}(n) \end{aligned}$$

where

- $p^{+}(n)$ = right-going *longitudinal pressure*
- $p^{-}(n)$ = left-going longitudinal pressure
- $u^{\pm}(n)$ = left and right-going *volume-velocity waves*
- wave impedance is

$$\boxed{R_{\mathsf{T}} = \frac{\rho c}{A}} \qquad \text{(Acoustic Tubes)}$$

  where
  - $\rho$ = mass per unit volume of air
  - $c$ = sound speed in air
  - $A$ = cross-sectional area of tube
- For *particle velocity*, wave impedance $= R_0 = \rho c$
- Particle velocity is appropriate in open air, while volume velocity is used for acoustic tubes

## Power Waves

Physically,

$$\begin{aligned}\text{Power} &= \text{Work/Time}\\ &= \text{Force} \times \text{Distance/Time}\\ &= \text{Force} \times \text{Velocity}\end{aligned}$$

*Traveling power waves:*

$$\begin{aligned}\mathcal{P}^+(n) &\stackrel{\Delta}{=} \quad f^+(n)v^+(n)\\ \mathcal{P}^-(n) &\stackrel{\Delta}{=} -\, f^-(n)v^-(n)\end{aligned}$$

From "Ohm's law" $f^+ = Rv^+$ and $f^- = -Rv^-$, we have

$$\begin{aligned}\mathcal{P}^+(n) &= R[v^+(n)]^2 = \frac{[f^+(n)]^2}{R}\\ \mathcal{P}^-(n) &= R[v^-(n)]^2 = \frac{[f^-(n)]^2}{R}\end{aligned}$$

Note that both $\mathcal{P}^+$ and $\mathcal{P}^-$ are *nonnegative*

*Summing* traveling powers gives net power flow

$$\mathcal{P}(t_n, x_m) \stackrel{\Delta}{=} \mathcal{P}^+(n-m) + \mathcal{P}^-(n+m)$$



## Energy Density Waves

Energy density = potential + kinetic energy densities:

$$W(t,x) \stackrel{\Delta}{=} \underbrace{\frac{1}{2}Ky'^2(t,x)}_{\text{potential}} + \underbrace{\frac{1}{2}\epsilon\dot{y}^2(t,x)}_{\text{kinetic}}$$

Sampled wave energy density can be expressed as

$$W(t_n, x_m) \stackrel{\Delta}{=} W^+(n-m) + W^-(n+m)$$

where

$$\begin{aligned}W^+(n) &= \frac{\mathcal{P}^+(n)}{c} = \frac{f^+(n)v^+(n)}{c} = \epsilon\left[v^+(n)\right]^2 = \frac{\left[f^+(n)\right]^2}{K}\\ W^-(n) &= \frac{\mathcal{P}^-(n)}{c} = -\frac{f^-(n)v^-(n)}{c} = \epsilon\left[v^-(n)\right]^2 = \frac{\left[f^-(n)\right]^2}{K}\end{aligned}$$

*Total wave energy* in string of length $L$:

$$\mathcal{E}(t) = \int_{x=0}^{L} W(t,x)dx \approx \sum_{m=0}^{L/X-1} W(t,x_m)X$$



## Root-Power Waves

Wave variables *normalized* to *square root* of power carried:

$$\begin{aligned}\tilde{f}^+ &\stackrel{\Delta}{=} f^+/\sqrt{R} \qquad & \tilde{f}^- &\stackrel{\Delta}{=} f^-/\sqrt{R}\\ \tilde{v}^+ &\stackrel{\Delta}{=} v^+\sqrt{R} \qquad & \tilde{v}^- &\stackrel{\Delta}{=} v^-\sqrt{R}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\mathcal{P}^+ &= \quad f^+v^+ &&= \tilde{f}^+\tilde{v}^+\\ &= \quad R(v^+)^2 &&= (\tilde{v}^+)^2\\ &= (f^+)^2/R &&= (\tilde{f}^+)^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{P}^- &= -f^-v^- &&= -\tilde{f}^+\tilde{v}^+\\ &= \quad R(v^-)^2 &&= (\tilde{v}^-)^2\\ &= (f^-)^2/R &&= (\tilde{f}^-)^2\end{aligned}$$

- Normalized wave variables $\tilde{f}^\pm$ and $\tilde{v}^\pm$ behave physically like force and velocity waves
- Either can be squared to obtain signal power
- Dynamic range is normalized in $L_2$ sense
- Driving a normalized waveguide network with unit variance white noise gives signal power equal to $1$ throughout the network
- Time-varying wave impedances do not cause "parametric amplification"