MUS420 Lecture
Digitizing Traveling Waves in Vibrating Strings

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## Outline

- Ideal vibrating string
- Traveling-wave solution
- Sampled traveling waves

## Ideal Vibrating String Model

We know already how to model a string as a *bidirectional delay line* with

- inverting reflecting terminations (for displacement)
- filters for loss and dispersion
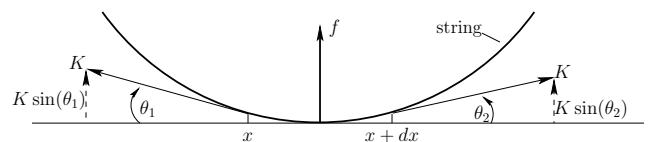- outputs as sums of traveling-wave components

This model is based on *traveling waves* and the *superposition* of traveling waves as *experimental fact*. In such a model, sound-speed must be measured experimentally.

We now take our string model to the next level based on the *physics* of ideal strings:

- Sound speed becomes a *predicted* quantity
- The very useful concept of *wave impedance* is derived

## Ideal String Physics



### Wave Equation

$$Ky'' = \epsilon \ddot{y}$$

$K \triangleq$ string tension $\qquad y \triangleq y(t,x)$
$\epsilon \triangleq$ linear mass density $\qquad \dot{y} \triangleq \frac{\partial}{\partial t}y(t,x)$
$y \triangleq$ string displacement $\qquad y' \triangleq \frac{\partial}{\partial x}y(t,x)$

### Newton's second law

$$\boxed{\text{Force} = \text{Mass} \times \text{Acceleration}}$$

### Assumptions

- Lossless
- Linear
- Flexible (no "Stiffness")
- Slope $y'(t,x) \ll 1$

## String Wave Equation Derivation



Force diagram for length $dx$ string element

Total upward force on length $dx$ string element:

$$
\begin{aligned}
f(x + dx/2) &= K\sin(\theta_1) + K\sin(\theta_2) \\
&\approx K\left[\tan(\theta_1) + \tan(\theta_2)\right] \\
&= K\left[-y'(x) + y'(x+dx)\right] \\
&\approx K\left[-y'(x) + y'(x) + y''(x)dx\right] \\
&= Ky''(x)dx
\end{aligned}
$$

Mass of length $dx$ string segment: $m = \epsilon\, dx$.

By Newton's law, $f = ma = m\ddot{y}$, we have

$$Ky''(t,x)dx = (\epsilon\, dx)\ddot{y}(t,x)$$

or

$$\boxed{Ky''(t,x) = \epsilon\ddot{y}(t,x)}$$

## Traveling-Wave Solution

**One-dimensional lossless wave equation:**

$$\boxed{Ky'' = \epsilon \ddot{y}}$$

Plug in *traveling wave to the right*:

$$y(t, x) = y_r(t - x/c)$$

$$\Rightarrow \quad y'(t, x) = -\frac{1}{c}\dot{y}(t, x)$$

$$y''(t, x) = \frac{1}{c^2}\ddot{y}(t, x)$$

- Given $c \triangleq \sqrt{K/\epsilon}$, the wave equation is satisfied for *any shape traveling to the right at speed $c$* (but remember slope $\ll 1$)

- Similarly, any *left-going* traveling wave at speed $c$, $y_l(t + x/c)$, satisfies the wave equation (show)

- General solution to lossless, 1D, second-order wave equation:

$$y(t, x) = y_r(t - x/c) + y_l(t + x/c)$$

- $y_l(\cdot)$ and $y_r(\cdot)$ are arbitrary twice-differentiable functions (slope $\ll 1$)

- **Important point:** Function of two variables $y(t, x)$ is replaced by two functions of a single (time) variable $\Rightarrow$ *reduced computational complexity*.

- Published by d'Alembert in 1747 (wave equation itself introduced in same paper)

## Infinitely long string plucked simultaneously at three points marked 'p'
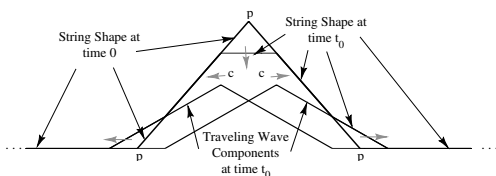


- Initial displacement = sum of two identical triangular pulses

- At time $t_0$, traveling waves centers are separated by $2ct_0$ meters

- String is not moving where the traveling waves overlap at same slope.

- Nelson Lee's Animation[1]

- Travis Skare's Interactive Animation[2]

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[1] http://ccrma.stanford.edu/~jos/rsadmin/TravellingWaveApp.swf
[2] https://ccrma.stanford.edu/~travissk/dwgdemo/

## Sampled Traveling Waves in a String

For discrete-time simulation, we must *sample* the traveling waves

- Sampling interval $\triangleq T$ seconds

- Sampling rate $\triangleq f_s$ Hz $= 1/T$

- Spatial sampling interval $\triangleq X$ m/s $\triangleq cT$ $\Rightarrow$ *systolic grid*

For a vibrating string with length $L$ and fundamental frequency $f_0$,

$$c = f_0 \cdot 2L \qquad \left(\frac{\text{meters}}{\text{sec}} = \frac{\text{periods}}{\text{sec}} \cdot \frac{\text{meters}}{\text{period}}\right)$$

so that

$$X = cT = (f_0 2L)/f_s = L[f_0/(f_s/2)]$$

Thus, the number of *spatial samples* along the string is

$$\boxed{L/X = (f_s/2)/f_0}$$

or

$$\boxed{\text{Number of spatial samples} = \text{Number of string harmonics}}$$

**Examples:**

- Spatial sampling interval for CD-quality digital model of Les Paul electric guitar (strings $\approx 26$ inches)

  - $X = Lf_0/(f_s/2) = L82.4/22050 \approx 2.5$ mm for low E string
  - $X \approx 10$ mm for high E string (two octaves higher and the same length)
  - Low E string: $(f_s/2)/f_0 = 22050/82.4 = 268$ harmonics (spatial samples)
  - High E string: 67 harmonics (spatial samples)

- Number of harmonics = number of oscillators required in *additive synthesis*

- Number of harmonics = number of two-pole filters required in *subtractive, modal,* or *source-filter decomposition synthesis*

- Digital waveguide model needs only *one delay line* (length $2L$)

**Examples (continued):**

- Sound propagation in *air*:

  - Speed of sound $c \approx 331$ meters per second
  - $X = 331/44100 = 7.5$ mm
  - Spatial sampling rate $= \nu_s = 1/X = 133$ samples/m
  - Sound speed in air is *comparable* to that of transverse waves on a guitar string (faster than some strings, slower than others)
  - Sound travels much faster in most solids than in air
  - Longitudinal waves in strings travel faster than transverse waves
    * typically an order of magnitude faster

# Sampled Traveling Waves in any Digital Waveguide

$$
\begin{aligned}
x &\to x_m = mX \\
t &\to t_n = nT
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
y(t_n, x_m) &= y_r(t_n - x_m/c) + y_l(t_n + x_m/c) \\
&= y_r(nT - mX/c) + y_l(nT + mX/c) \\
&= y_r\left[(n - m)T\right] + y_l\left[(n + m)T\right] \\
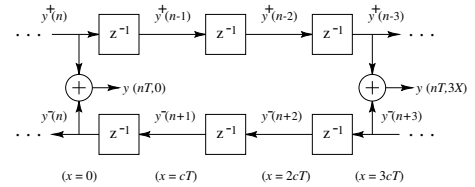&= y^+(n - m) + y^-(n + m)
\end{aligned}
$$

when $X = cT$, where we defined

$$
y^+(n) \triangleq y_r(nT) \qquad y^-(n) \triangleq y_l(nT)
$$

- "+" superscript $\implies$ *right-going*

- "$-$" superscript $\implies$ *left-going*

- $y_r\left[(n - m)T\right] = y^+(n - m) =$ output of $m$-sample delay line with input $y^+(n)$

- $y_l\left[(n + m)T\right] \triangleq y^-(n + m) = $ *input* to an $m$-sample delay line whose *output* is $y^-(n)$

**Lossless digital waveguide with observation points at $x = 0$ and $x = 3X = 3cT$**



- Recall:

$$
y(t, x) = y^+\left(\frac{t - x/c}{T}\right) + y^-\left(\frac{t + x/c}{T}\right)
$$

$$
\downarrow
$$

$$
y(nT, mX) = y^+(n - m) + y^-(n + m)
$$

- *Position* $x_m = mX = mcT$ is *eliminated* from the simulation

- Position $x_m$ remains laid out from left to right

- Left- and right-going traveling waves must be *summed* to produce a *physical* output

$$
y(t_n, x_m) = y^+(n - m) + y^-(n + m)
$$

- Similar to *ladder* and *lattice digital filters*

**Important point:** Discrete time simulation is *exact* at the sampling instants, to within the numerical precision of the samples themselves.

To avoid *aliasing* associated with sampling:

- Require all initial waveshapes be *bandlimited* to $(-f_s/2, f_s/2)$
- Require all external driving signals be similarly bandlimited
- Avoid nonlinearities or keep them "weak"
- Avoid time variation or keep it slow
- Use plenty of oversampling and lowpass filtering with rapid high-frequency roll-off in severely nonlinear and/or time-varying cases
- Prefer "feed-forward" over "feed-back" around nonlinearities and/or modulations when possible

Interactive simulation of a vibrating string:
`http://www.colorado.edu/physics/phet/simulations/-stringwave/stringWave.swf`

## Digital Waveguide Plucked-String Model Using Initial Conditions
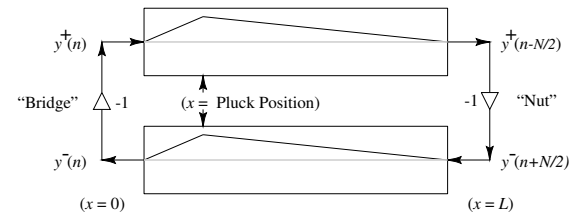


Initial conditions for the ideal plucked string.

- Amplitude of each traveling-wave $= 1/2$ initial string displacement.
- Sum of the upper and lower delay lines $=$ initial string displacement.

## Other Wave Variables (Wave-Impedance Preview)

**Transverse Velocity Waves:**

$$v^+(n) \triangleq \dot{y}^+(n)$$
$$v^-(n) \triangleq \dot{y}^-(n)$$

**Wave Impedance (we'll derive later):**

$$\boxed{R = \sqrt{K\epsilon} = \frac{K}{c} = \epsilon c}$$

**Force Waves:**

$$f^+(n) \triangleq R\,v^+(n)$$
$$f^-(n) \triangleq -R\,v^-(n)$$

**Ohm's Law for Traveling Waves:**

$$\boxed{\begin{aligned} f^+(n) &= \phantom{-}R\,v^+(n) \\ f^-(n) &= -\,R\,v^-(n) \end{aligned}}$$

## Acoustic Plane Waves

**Pressure Plane Waves:**

$$p^+(n) \triangleq R_a\,u^+(n)$$
$$p^-(n) \triangleq -R_a\,u^-(n)$$

where $u^+, u^-$ are
**Longitudinal Particle-Velocity Waves**

**Ohm's Law for Traveling Acoustic Plane Waves:**

$$\boxed{\begin{aligned} p^+(n) &= \phantom{-}R_a u^+(n) \\ p^-(n) &= -\,R_a u^-(n) \end{aligned}}$$

where

$$\boxed{R_a = \rho c}$$

is the wave impedance of air in terms of mass density $\rho$ (kg/m$^3$) and sound speed $c$.

# Acoustic Tubes

In acoustic tubes, we again work with
**Pressure Plane Waves:**

$$p^+(n) \triangleq R_{\tau} U^+(n)$$
$$p^-(n) \triangleq -R_{\tau} U^-(n)$$

However, now $U^+, U^-$ are
**Longitudinal Volume-Velocity Waves:**

$$U^+(n) \triangleq A\, u^+(n)$$
$$U^-(n) \triangleq A\, u^-(n)$$

where $A$ is the cross-sectional area of the tube. In an acoustic tube, it is volume velocity that is conserved from one tube section to the next.

**Ohm's Law for Traveling Plane Waves in an Acoustic Tube:**

$$\boxed{\begin{aligned} p^+(n) &= \phantom{-}\,R_{\tau} U^+(n) \\ p^-(n) &= -\,R_{\tau} U^-(n) \end{aligned}}$$

where

$$\boxed{R_{\tau} = \frac{\rho c}{A}}$$

is the wave impedance of air in terms of mass density $\rho$, sound speed $c$, and tube cross-section area $A$.