MUS420/EE367A Lecture 6D
Plane-Wave Scattering at an Angle

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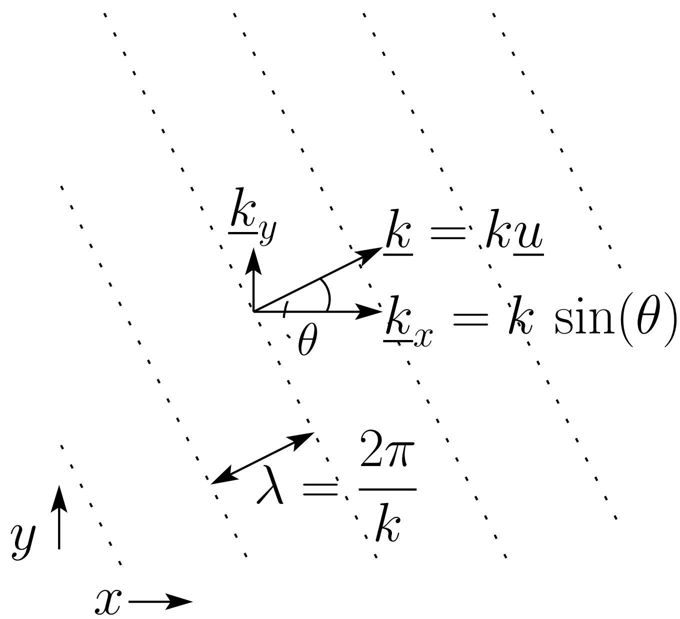
Outline

- Plane Wave at an Angle
- Plane-Wave Scattering at an Impedance Discontinuity
- Reflection and Refraction
- Evanescent Field due to Total Internal Reflection
- Plane-Wave Scattering at an Angle
- Imaginary wavenumbers

Plane Wave at an Angle



wave crests of the sinusoidal traveling plane wave
$p(t, \underline{x}) = \cos\left(\omega t - \underline{k}^T \underline{x}\right)$

Planar pressure-wave traveling in an arbitrary direction:

$$p(t, \underline{x}) = \cos\left(\omega t - \underline{k}^T \underline{x}\right), \quad \underline{x} \in \mathbf{R}^3$$

where $\underline{k}$ = *vector wavenumber*:

$$\underline{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = k \begin{bmatrix} k_x/k \\ k_y/k \\ k_z/k \end{bmatrix} \stackrel{\Delta}{=} k \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix} \stackrel{\Delta}{=} k\,\underline{u},$$

where

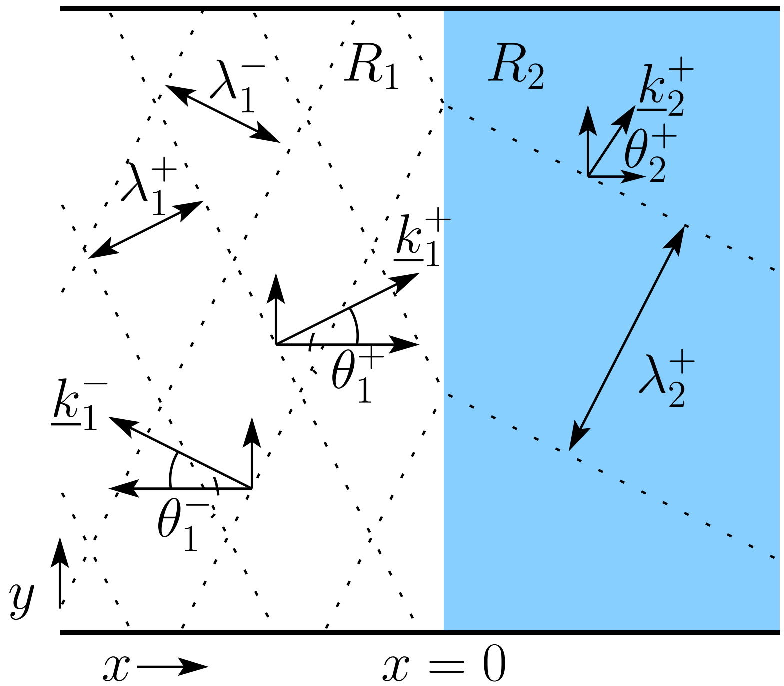
- $\underline{u} =$ (unit) vector of *direction cosines*
- $k = \frac{2\pi}{\lambda} =$ (scalar) *wavenumber* along travel direction

Thus, the vector wavenumber $\underline{k} = k\,\underline{u}$ contains

- wavenumber in its magnitude $k = \|\,\underline{k}\,\|$
- travel direction in its orientation $\underline{u} = \underline{k}/k$

Note: wavenumber units are *radians per meter*
(spatial radian frequency)

Oblique Plane-Wave Scattering



By continuity, waves must agree on boundary plane:

$$\left\langle \underline{k}_1^+, \underline{r} \right\rangle = \left\langle \underline{k}_1^-, \underline{r} \right\rangle = \left\langle \underline{k}_2^+, \underline{r} \right\rangle$$

where $\underline{r} = (0, y, z)$ denotes any vector in the boundary plane. Thus, at $x = 0$ we have

$$k_{1y}^+ \, y + k_{1z}^+ \, z = k_{1y}^- \, y + k_{1z}^- \, z = k_{2y}^+ \, y + k_{2z}^+ \, z$$

If the incident wave is constant along z, then $k_{1z}^+ = 0$, requiring $k_{1z}^- = k_{2z}^+ = 0$, leaving

$$k_{1y}^+ \, y = k_{1y}^- \, y = k_{2y}^+ \, y$$

or

$$\boxed{k_1 \sin(\theta_1^+) = k_1 \sin(\theta_1^-) = k_2 \sin(\theta_2^+)}$$

Reflection and Refraction

Above we derived

$$k_1 \sin(\theta_1^+) = k_1 \sin(\theta_1^-) = k_2 \sin(\theta_2^+)$$

The first equality implies

$$\boxed{\theta_1^+ = \theta_1^-}$$

(Angle of incidence equals angle of reflection)

Let c_i denote the phase velocity in wave impedance R_i:

$$c_i = \frac{\omega}{k_i}, \quad i = 1, 2$$

In impedance R_2, we have in particular

$$\omega^2 = c_2^2 k_2^2 = c_2^2 \left[(k_{2x}^+)^2 + (k_{2y}^+)^2 \right]$$

Solving for k_{2x}^+ gives

$$k_{2x}^+ = \sqrt{\frac{\omega^2}{c_2^2} - (k_{2y}^+)^2} = \sqrt{\frac{\omega^2}{c_2^2} - k_2^2 \sin^2(\theta_2^+)}$$

Since $k_1 \sin(\theta_1^+) = k_2 \sin(\theta_2^+)$ from above,

$$k_{2x}^+ = \sqrt{\frac{\omega^2}{c_2^2} - k_1^2 \sin^2(\theta_1^+)} = \sqrt{\frac{\omega^2}{c_2^2} - \frac{\omega^2}{c_1^2} \sin^2(\theta_1^+)}$$

We have derived

$$\boxed{k_{2x}^{+} = \frac{\omega}{c_2}\sqrt{1 - \frac{c_2^2}{c_1^2}\sin^2(\theta_1^{+})}}$$

We earlier established $k_{2y}^{+} = k_{1y}^{+}$.

- This describes the *refraction* of the plane wave as it passes through the impedance-change boundary.
- Refraction angle depends on ratio of phase velocities c_2/c_1.
- This ratio is often called the *index of refraction*:

  $$n \triangleq \frac{c_2}{c_1}$$

  and the relation $k_1 \sin(\theta_1^{+}) = k_2 \sin(\theta_2^{+})$ is called *Snell's Law* (of refraction).

Evanescent Field due to Total Internal Reflection

Note that if $c_1 < c_2 |\sin(\theta_1^+)|$, the horizontal component of the wavenumber in medium 2 becomes *imaginary*:

- Acoustic field in medium 2 is *"evanescent"*
- Wave in medium 1 undergoes "*total internal reflection*"
- No power travels from medium 1 to medium 2
- Evanescent field decays exponentially to the right
- "Tunneling" possible given medium 3 in which wave propagation resumes
- Reference: **Vibrations and Waves in Physics** by I.G. Main, Cambridge University Press, 1978.

What does it mean to have an imaginary wavenumber?

$$
\begin{aligned}
p(t, \underline{x}) &= \cos\left(\omega t - \underline{k}^T \underline{x}\right) = \text{re}\left\{e^{j(\omega t - k_x x - k_y y)}\right\} \\
&= \text{re}\left\{e^{j(\omega t - k_y y)} e^{-j k_x x}\right\}, \quad (\text{let } k_x \stackrel{\Delta}{=} -j\kappa_x) \\
&\stackrel{\Delta}{=} \text{re}\left\{e^{j(\omega t - k_y y)} e^{-k_x x}\right\} \\
&= e^{-k_x x} \cos(\omega t - k_y y)
\end{aligned}
$$

- An imaginary wavenumber corresponds to an exponential decay
- Sign of κ_x is chosen to match boundary conditions at the plane
- Time dependence applies to *all* points to the right of the boundary (no "propagation")