MUS420 Lecture
Nonlinear Elements

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June 27, 2020

Outline

- Nonlinearities in musical instrument models
- Memoryless nonlinearities
- Bandwidth expansion and aliasing due to nonlinearities

Nonlinearities

Many musical instrument models require *nonlinear elements*:

- Amplifier distortion (electric guitar)
- Reed model (woodwinds)
- Bowed string contact friction

Since a nonlinear element generally *expands signal bandwidth*, it can cause *aliasing* in a discrete-time implementation.

In the above examples, the nonlinearity also appears inside a *feedback loop*. This means the bandwidth expansion *compounds* over time, causing more and more aliasing.

Memoryless Nonlinearities

Memoryless or *instantaneous* nonlinearities are the simplest and most commonly implemented form of nonlinear element:

$$y(n) = f(x(n))$$

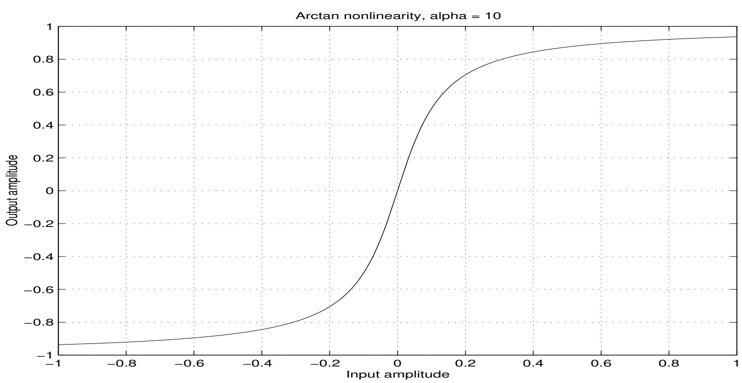
excluding the special case $f(x) = \alpha x$ which defines a simple *linear gain* of α.

Example: Arctan Nonlinearity

An example of an *invertible* memoryless nonlinearity is the *arctangent* mapping:

$$f(x) = \frac{2}{\pi}\arctan(\alpha x), \quad x \in [-1, 1]$$

where normally $\alpha \gg 1$.

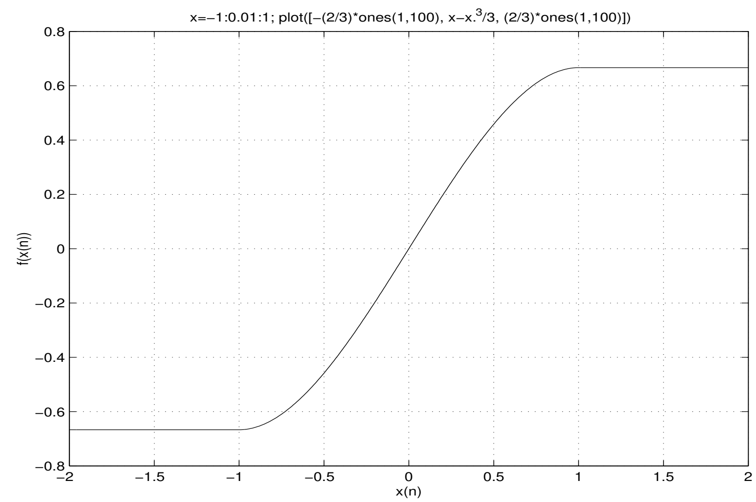


Cubic Soft Clipper

We used the cubic soft-clipper in simulating amplifier distortion:

$$f(x) = \begin{cases} -\frac{2}{3}, & x \le -1 \\ x - \frac{x^3}{3}, & -1 \le x \le 1 \\ \frac{2}{3}, & x \ge 1 \end{cases}$$

It is non-invertible when driven into "hard clipping".



Series Expansions

Any "smooth" function $f(x)$ can be expanded as a Taylor series expansion:

$$f(x) = f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{1 \cdot 2}x^2 + \frac{f'''(0)}{1 \cdot 2 \cdot 3}x^3 + \cdots ,$$

Arctangent example:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Note that all even-order terms are zero, because $\arctan(x)$ is an *odd function* of x.

Bandwidth Expansion

The *series expansion* of a memoryless nonlinearity gives a useful handle on *aliasing*.

Square Law

The "gentlest" nonlinearity is *quadratic*:

$$y(n) = x(n) + \alpha x^2(n)$$

The Fourier transform of the output signal is easily found using the dual of the convolution theorem:

$$Y(\omega) = X(\omega) + \alpha(X * X)(\omega)$$

where "$*$" denotes convolution.

In general, the bandwidth of $X * X$ is *double* that of X.

More generally,

$$x^k(n) \;\longleftrightarrow\; \underbrace{(X * X * \cdots * X)}_{k \text{ times}}(\omega)$$

so that the spectral bandwidth of $x^k(n)$ is k times that of $x(n)$, in general.

Arctangent Nonlinearity

Since the series expansion of the arctangent nonlinearity is

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

bandwidth expansion is *infinite* (in continuous time).

Cubic Soft-Clipper

The cubic soft-clipper, like any polynomial nonlinearity, is defined directly by its series expansion:

$$f(x) = \begin{cases} -\frac{2}{3}, & x \le -1 \\ x - \frac{x^3}{3}, & -1 \le x \le 1 \\ \frac{2}{3}, & x \ge 1 \end{cases}$$

In the absence of hard-clipping ($|x| \le 1$), bandwidth expansion is limited to a factor of *three*.

This is the slowest aliasing rate obtainable for an *odd* nonlinearity.

Practical Advice

- Verify that aliasing sounds bad before getting rid of it
- Aliasing (bandwidth expansion) is reduced by smoothing the "corner" in the clipping nonlinearity
- Consider a healthy *oversampling factor* for nonlinear subsystems
- Make sure there is adequate lowpass filtering in a feedback loop containing a nonlinearity

Example: Cubic Nonlinearity in a Feedback Loop:

- 3X oversampling
  (2X suffices for full-band audio, since aliasing into the guard-band above 20 kHz is inaudible)
- Lowpass filter to $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$ after the nonlinearity
- Optionally downsample by 3 after LPF and upsample by 3 before nonlinearity