MUS420/EE367A Lecture 7
Mass Striking a String (Simplified Piano Model)

Julius O. Smith III (jos@ccrma.stanford.edu)
Center for Computer Research in Music and Acoustics (CCRMA)
Department of Music, Stanford University
Stanford, California 94305

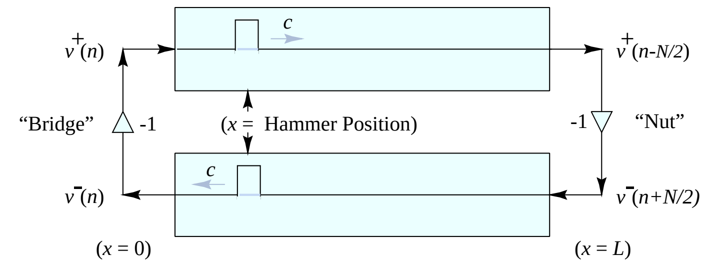
March 1, 2007

Outline

- Physical model
- Equivalent Circuit
- Impedance analysis
- Digital waveguide model

Ideal String Struck by a Mass

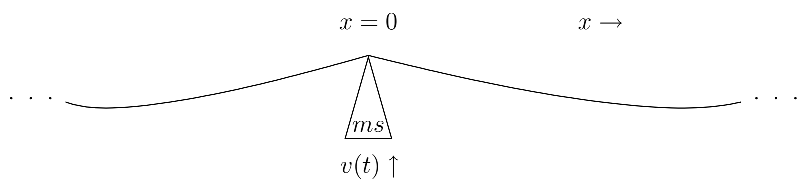
Previously (and in the text), the ideal struck string was modeled as a simple initial velocity distribution along the ideal string:



- Hammer momentum transferred to string at time 0.
- Hammer gone after time 0 ("elastic collision")

We now derive a model in which the hammer strikes the string and remains in contact ("inelastic collision")

Physical Model (Ideal String Struck by a Mass)



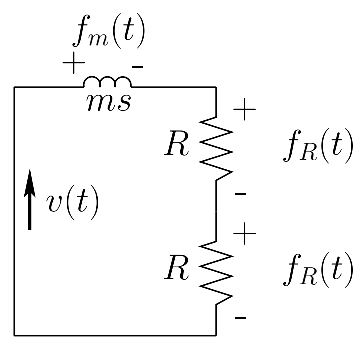
- String infinitely long — no gravity or air
- Wave impedance R
- Mass m strikes string at $x = 0$ (a *single point*!)
- Collision speed v_0
- Horizontal motion neglected

At time 0, our model *switches* from

1. mass-in-flight and ideal string, to
2. two ideal strings joined by mass m at $x = 0$

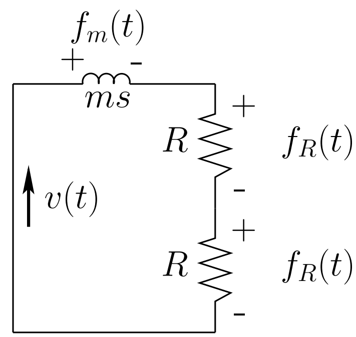
Note that the mass and two string-segment impedances are in *series* because they all move together

Equivalent Circuit (Ideal String Struck by a Mass)



- Mass = *inductor* of m Henrys (impedance ms)
- String endpoint = *resistor* of R Ohms (impedance R)
- *One common velocity* for all three elements ⇒ *series*
- Series order is arbitrary
- Note that string wave impedance appears *twice*
- "Polarity" (reference direction) for each element defines "positive" current for that element: Current is positive when flowing from $+$ to $-$
- Positive current corresponds to a positive voltage drop

Equivalent Circuit Analysis



Kirchoff's Loop Rule:

> The sum of voltages ("forces") around any series loop is zero

So

$$f_m(t) + f_R(t) + f_R(t) = 0$$

Laplace transform:

$$F_m(s) + 2F_R(s) = 0,$$

Laplace Transform Analysis

From the equivalent circuit, we derived

$$F_m(s) + 2F_R(s) = 0,$$

where

- $F_m(s)$ = Laplace transform of upward mass force
- $F_R(s)$ = Laplace transform of upward force on each string segment
- Recall:

$$F_m(s) \stackrel{\Delta}{=} \mathcal{L}_s\{f_m\} \stackrel{\Delta}{=} \int_0^\infty f_m(t) e^{-st} dt$$

Equations of motion:

- String:

$$\boxed{f_R(t) = R\,v(t) \quad \leftrightarrow \quad F_R(s) = R\,V(s)}$$

  where $V(s) \stackrel{\Delta}{=} \mathcal{L}_s\{v\}$ is the Laplace transform of $v(t)$
- Mass:

$$\boxed{f_m(t) = m\,a(t) \;=\; m\,\dot{v}(t) \;\leftrightarrow\; F_m(s) = m\,[s\,V(s) - v_0]}$$

  (recall the Laplace-transform differentiation theorem)

Motion from Initial Conditions

From the equivalent circuit, we derived

$$F_m(s) + 2F_R(s) = 0$$

where

$$\begin{aligned} F_R(s) &= R\,V(s) \\ F_m(s) &= ms\,V(s) - m\,v_0 \end{aligned}$$

Substituting gives

$$m\,s\,V(s) - m\,v_0 + 2R\,V(s) = 0$$

Solving for $V(s)$ gives

$$\boxed{V(s) = \frac{m\,v_0}{ms + 2R}}$$

Since

$$e^{-at}u(t) \;\longleftrightarrow\; \frac{1}{s+a}$$

(where $u(t)$ = unit step function), we find the velocity of the mass-string contact point to be

$$\boxed{v(t) = v_0\,e^{-\frac{2R}{m}t}, \quad t \geq 0}$$

Observations

We derived that the velocity of the mass-string contact point is

$$v(t) = v_0\,e^{-\frac{2R}{m}t}, \quad t \geq 0$$

- At time zero the mass velocity is v_0, as it must be
- Velocity decays exponentially to zero with time-constant $m/2R$
- Decay rate is proportional to R/m
- Since $R = \sqrt{K\epsilon}$, decay is faster if either string tension K or mass-density ϵ is increased
- String *displacement* at $x = 0$ is given by the integral of transverse velocity:

  $$y(t,0) = \int_0^t v(\tau)\,d\tau = v_0\,\frac{m}{2R}\left[1 - e^{-\frac{2R}{m}t}\right]$$

  where we define initial displacement to be $y(0,0) = 0$

Momentum Transfer

- Force applied to the two string endpoints by the mass is given by $f_m(t) = 2Rv(t)$
- From Newton's Law, $f = ma = m\dot{v}$, momentum mv is the *time integral* of force:

$$\int_0^t f_m(\tau)\, d\tau \;=\; 2R \int_0^t v(\tau)\, d\tau \;=\; m\, v_0 \left(1 - e^{-\frac{2R}{m}t}\right)$$

  - Thus, momentum delivered to the string by the mass starts at zero and grows as a relaxing exponential to mv_0 as $t \to \infty$
  - Inelastic mass-string collision is *not* an instantaneous momentum transfer
  - Mass momentum transfers exponentially over time
  - This is why contact width can be zero
- In a real piano, the hammer, which strikes in an upward direction, falls away from the string a short time after collision, but it may remain in contact with the string for a substantial fraction of a period

Looking at the Mass from the String

Model:

- Mass divides string into two *segments*
- String segment model = *digital waveguide*
- Mass model = *scattering junction*

Force Equation:

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$

where

- $f_{1m}(t)$ = force applied by string-segment 1 to the mass
- $f_{2m}(t)$ = force applied by string-segment 2 to the mass
- $f_m(t)$ = inertial force applied by mass to both string endpoints
- Force is positive in the "up" direction

Force Wave Variables

Force Equation:

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$

Traveling-Wave Decomposition of String Force:

$$\begin{aligned} f_1(t,x) &= f_1^+(t - x/c) + f_1^-(t + x/c) \\ f_2(t,x) &= f_2^+(t - x/c) + f_2^-(t + x/c) \end{aligned}$$

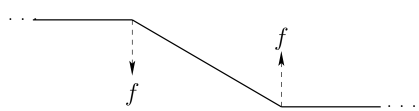
String Force:

$$f(t,x) \stackrel{\Delta}{=} -Ky'(t,x)$$

where

- K = string tension
- y' = string slope

Note that string force *pulls up to the right*.
That is, a string segment with negative slope pulls "up" to the right and "down" to the left:



In the present problem, our force equation

$$f_m(t) + f_{1m}(t) + f_{2m}(t) = 0,$$

becomes, in terms of mass inertial force and string forces:

$$m\dot{v}(t) + K\,y_1'(t,0) - K\,y_2'(t,0) = 0$$

or, using string force-wave notation:

$$\boxed{f_m(t) - f_1(t) + f_2(t) = 0}$$

These force relations can be checked individually:

- For string 1,

$$m\dot{v}(t) + K\,y_1'(t,0) = 0$$

$\Longrightarrow$ positive slope on left accelerates mass down

- For string 2,

$$m\dot{v}(t) - K\,y_2'(t,0) = 0$$

$\Longrightarrow$ positive slope on right accelerates mass up

Digital Waveguide Model

Above we derived

$$f_m(t) - f_1(t) + f_2(t) = 0$$

We now perform the traveling-wave decompositions

$$\begin{aligned} f_1 &= f_1^+ + f_1^- \\ f_2 &= f_2^+ + f_2^- \end{aligned}$$

and apply the Ohm's law relations

$$\begin{aligned} f_i^+ &= Rv_i^+ \\ f_i^- &= -Rv_i^-, \quad i = 1, 2 \end{aligned}$$

to obtain a digital waveguide model

- By symmetry, the mass must look identical from either string segment 1 or 2
- Therefore, let $f_2^- \equiv 0$ and excite from string 1

In the Laplace domain, we have

$$\begin{aligned} 0 &= F_m - F_1 + F_2 \\ &= F_m - (F_1^+ + F_1^-) + F_2^+ \qquad \text{(since } F_2^- = 0\text{)} \end{aligned}$$

DWM, Cont'd

We just derived

$$F_m(s) - F_1^+(s) + F_1^-(s) + F_2^+(s) = 0$$

(no incoming wave from string 2)

Substitute

$$\begin{aligned} F_i^+(s) &= RV_i^+(s) \\ F_i^-(s) &= -RV_i^-(s) \\ F_m(s) &= msV(s) \end{aligned}$$

to obtain

$$msV - RV_1^+ + RV_1^- + RV_2^+ = 0$$

We always have $V = V_1^+ + V_1^- = V_2^+ + V_2^-$
(series combination)
Since $V_2^- = 0$, we have $V_2^+ = V$, and

$$\begin{aligned} (R + ms)V - RV_1^+ + RV_1^- &= 0 \\ \Rightarrow \quad (R + ms)(V_1^+ + V_1^-) - RV_1^+ + RV_1^- &= 0. \end{aligned}$$

Solving for the *velocity reflection transfer function* (or velocity *reflectance*) of the mass from string 1 gives

$$\boxed{\rho_1^v \triangleq \frac{V_1^-}{V_1^+} = -\frac{ms}{ms + 2R}}$$

By physical symmetry, the mass looks the same from string 2:

$$\rho_2^v \triangleq \frac{V_2^+}{V_2^-} = \rho_1^v = -\frac{ms}{ms + 2R}$$

(reflectance of a mass m at the end of a string of wave impedance R)

Limiting Behavior:

- When $m = 0$, reflectance is zero (no reflected wave)
- When $m \to \infty$, $\rho_1^v \to -1$ (rigid termination)

Simplified Impedance Analysis

The above results are quickly derived from the general reflection-coefficient for force waves (or voltage waves, pressure waves, etc.):

$$\rho = \frac{R_2 - R_1}{R_2 + R_1} = \frac{\text{Impedance Step}}{\text{Impedance Sum}}$$

where $\rho =$ reflection coefficient of impedance R_2 as "seen" from impedance R_1

When a force wave crosses from impedance R_1 to R_2 it splits into

1. a reflected wave $f^- = \rho f^+$ in R_1, and
2. a transmitted wave $(1 + \rho)f^+$ in R_2

Therefore, a velocity wave v^+ splits into

1. reflected wave $v^- = -\rho v^+$ and
2. transmitted wave $(1 - \rho)v^+$

These relations are of course unchanged in the Laplace domain, by linearity of the Laplace transform.

Mass Reflectance

In the mass-string-collision problem, we can immediately write down the *force reflectance* of the mass:

$$\hat{\rho}(s) = \frac{R_{\text{mass+string}} - R_{\text{string}}}{R_{\text{mass+string}} + R_{\text{string}}} = \frac{(ms+R)-R}{(ms+R)+R} = \frac{ms}{ms+2R}$$

The *velocity reflectance* is simply $-\hat{\rho}(s)$, since

$$\hat{\rho}(s) \stackrel{\Delta}{=} \frac{F^-}{F^+} = \frac{-RV^-}{RV^+} = -\frac{V^-}{V^+}$$

- Simplified impedance analysis is a nice shortcut
- Be careful to correctly identify the impedance jump

Mass Transmittance

Force transmittance is similarly derived:

$$\hat{\tau}_f(s) \stackrel{\Delta}{=} \frac{F}{F^+} = \frac{F^+ + F^-}{F^+} = 1 + \frac{F^-}{F^+} = 1 + \hat{\rho}(s)$$

as is *velocity transmittance*:

$$\hat{\tau}_v(s) \stackrel{\Delta}{=} \frac{V}{V^+} = \frac{V^+ + V^-}{V^+} = 1 + \frac{V^-}{V^+} = 1 - \hat{\rho}(s)$$

For the mass-on-string problem:

$$\hat{\tau}_f(s) \;=\; 1 + \hat{\rho}(s) = 1 + \frac{ms}{ms+2R} = 2\frac{ms+R}{ms+2R}$$

$$\hat{\tau}_v(s) \;=\; 1 - \hat{\rho}(s) = 1 - \frac{ms}{ms+2R} = \frac{2R}{ms+2R}$$

Limiting Behavior:

- For $m = 0$, both transmission filters become 1, as expected
- For $m = \infty$, $\hat{\tau}_v(s) \to 0$, which makes good physical sense (infinite mass = rigid termination)
- For $m = \infty$, $\hat{\tau}_f(s) \to 2$!
- Recall that signal power is force *times* velocity

Wave Scattering by a Mass on a String

We have derived the reflectance and transmittance of a mass m as seen from either string at impedance R. We can now derive the complete scattering relations:

For force waves, the outgoing waves are

$$\begin{aligned} F_1^-(s) &= \hat{\rho}(s)F_1^+(s) + \hat{\tau}_f(s)F_2^-(s) \\ F_2^+(s) &= \hat{\tau}_f(s)F_1^+(s) + \hat{\rho}(s)F_2^-(s) \end{aligned}$$

where the incoming waves are F_1^+ and F_2^-, and

$$\begin{aligned} \hat{\rho}(s) &= \frac{ms}{ms+2R} \quad \text{(force reflectance)} \\ \hat{\tau}_f(s) &= 1+\hat{\rho}(s) = \frac{2(ms+R)}{ms+2R} \quad \text{(force transmittance)} \end{aligned}$$

We may say that the mass creates a *dynamic scattering junction* on the string

One-Filter Scattering Junction

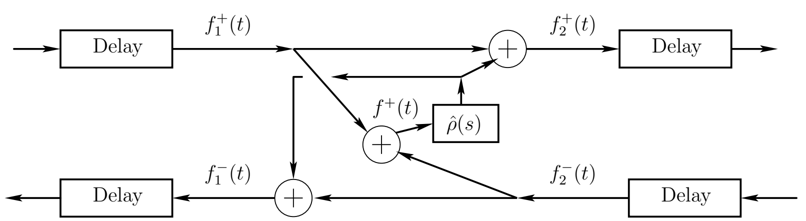
The scattering relations above can be said to be in "Kelly-Lochbaum form." The general relation $\hat{\tau}_f = 1 + \hat{\rho}$ can be used to simplify to a *one-filter dynamic scattering junction*

$$\begin{aligned} F_1^- &= \hat{\rho}F_1^+ + (1+\hat{\rho})F_2^- &= F_2^- + \hat{\rho}\cdot(F_1^+ + F_2^-) \\ F_2^+ &= (1+\hat{\rho})F_1^+ + \hat{\rho}F_2^- &= F_1^+ + \hat{\rho}\cdot(F_1^+ + F_2^-) \end{aligned}$$

The one-filter form follows from the observation that $\hat{\rho}\cdot(F_1^+ + F_2^-)$ appears in both computations, and therefore need only be implemented once:

$$\begin{aligned} F^+ &\stackrel{\Delta}{=} \hat{\rho}\cdot(F_1^+ + F_2^-) \\ F_1^- &= \hat{\rho}F_1^+ + (1+\hat{\rho})F_2^- &= F_2^- + F^+ \\ F_2^+ &= (1+\hat{\rho})F_1^+ + \hat{\rho}F_2^- &= F_1^+ + F^+ \end{aligned}$$

Signal Flow Diagram:



Digital Waveguide Model: Ideal String Struck by a Mass

We have derived an "analog waveguide model" for the ideal string struck by an ideal mass. We now *digitize* that model.

- Delays implemented using digital delay lines
- Mass reflectance digitize using the *bilinear transform*
- Force waves chosen

Mass Reflectance:

$$\hat{\rho}(s) = \frac{ms}{ms + 2R} = \frac{1}{1 + \frac{2R}{ms}} \tag{1}$$

Bilinear Transform:

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

where $T =$ sampling interval, yields

$$\hat{\rho}_d(z) = \frac{1}{1 + \frac{2R}{m}\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}} = g\frac{1 - z^{-1}}{1 - pz^{-1}}$$

Digitized Mass Reflectance

$$\boxed{\hat{\rho}_d(z) = g\frac{1 - z^{-1}}{1 - pz^{-1}}}$$

where

$$g \stackrel{\Delta}{=} \frac{1}{1 + \frac{RT}{m}} < 1$$

$$p \stackrel{\Delta}{=} \frac{1 - \frac{RT}{m}}{1 + \frac{RT}{m}} < 1$$

Thus, the mass reflectance is a one-pole, one-zero filter:

- Zero at dc
- Real pole close to dc at high sampling rates
- Unity gain at $f_s/2$
- Classic *dc-blocking filter*

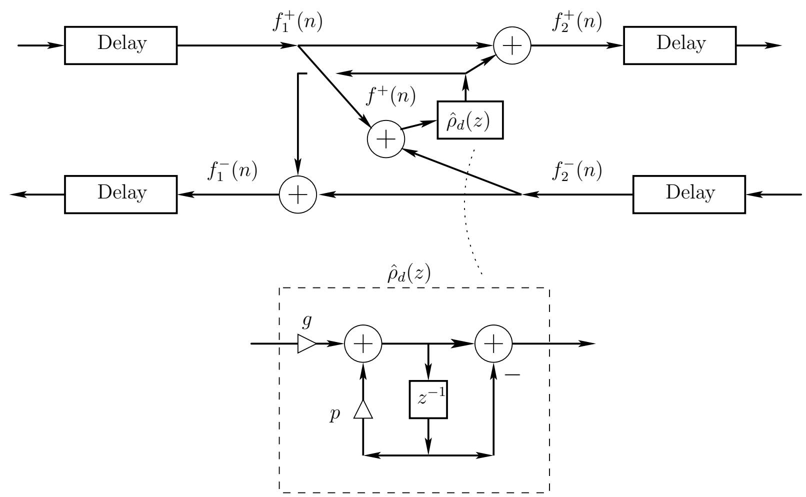
Why is the Mass Reflectance a DC-Blocker?

Physical intuition:

- The mass reflectance is zero at dc because Sufficiently slow force waves can freely move a mass of any finite size
- The reflectance is 1 at infinite frequency because there is no time for the mass to move before it is pushed in the opposite direction

In summary, the mass becomes a rigid termination at infinite frequency, and a free end (no termination) at zero frequency — a "dc blocker".

Final Digital Waveguide Model (Mass-Terminated String)



Final Notes

- Mass reflectance uses a *warped, unaliased* frequency axis (due to the bilinear transform) — can be viewed as a "Wave Digital Filter" (WDF) mass model
- Delay lines use an *unwarped, aliased* frequency axis (simple sampling) — "digital waveguide model"

- Frequency axes align well at low frequencies, or given sufficient oversampling
- Mass model is 1st-order in both analog and digital domains
- A higher order digital filter can be used for the mass to improve frequency response across the audio band