Phase Response

-Jon Dattorro

There is a misconception that not much in regard to phase can be visually determined from a pole/zero plot of an associated discrete signal or system [S&K,pg.262]. Here we will give rules that can be used to eyeball phase, given only a pole/zero plot.

Rule 1) ***Phase for real signals and phase response of real linear time-invariant systems is anti-symmetrical.***

Proof:

$$\operatorname{Arg}\left[H\left(e^{j\omega}\right)\right] = \operatorname{atan}\left(\frac{\operatorname{Im}\left[H\left(e^{j\omega}\right)\right]}{\operatorname{Re}\left[H\left(e^{j\omega}\right)\right]}\right) \qquad (1)$$

where $\operatorname{Arg}[\cdot]$ denotes the *principal value* of the phase, [Churchill] and where $\omega = 2\pi f T$, for T the sample period. The arctangent function $\operatorname{atan}(x)$ is an anti-symmetrical function (of an anti-symmetrical variable x). Real signals and systems have the property that $h[n] = h^*[n]$, hence they possess the property of Hermitian symmetry in their Fourier transforms; i.e.,

$$H\left(e^{-j\omega}\right) = H^*\left(e^{j\omega}\right) \qquad (1A)$$

The imaginary part of $H(e^{j\omega})$ is an anti-symmetrical function of frequency, while its real part is a symmetrical function. The variable $\operatorname{Im}[H(e^{j\omega})]/\operatorname{Re}[H(e^{j\omega})]$ is therefore anti-symmetrical, hence phase response is anti-symmetrical versus ω. Alternately, from Equ.(1A) we know that $|H(e^{-j\omega})|\, e^{j\operatorname{Arg}[H(e^{-j\omega})]} = |H(e^{j\omega})|\, e^{-j\operatorname{Arg}[H(e^{j\omega})]}$ that further shows that phase response must be anti-symmetrical. ◊

When using Equ.(1) to find system phase, keep in mind that your calculator probably employs the standard $\operatorname{atan}(v/u)$ function, hence returning phase in only two quadrants. For that reason, the $\operatorname{atan}(u,v)$ function was introduced into the C programming language, Matlab, *Mathematica*, and other languages to correct that problem.

$$\begin{aligned}\operatorname{atan}(u,v) &\equiv \operatorname{atan}\left(\frac{v}{u}\right) && ;u \geq 0\\ &\equiv \operatorname{atan}\left(\frac{v}{u}\right)+\pi && ;u<0,\ v\geq 0\\ &\equiv \operatorname{atan}\left(\frac{v}{u}\right)-\pi && ;u<0,\ v<0\end{aligned}$$

The phase correction of π observes the quadrant of the cartesian (u,v) coordinate system in which $\mathrm{Re}[H(e^{j\omega})]$ and $\mathrm{Im}[H(e^{j\omega})]$ reside.

Example 1

Suppose $H(e^{j\omega}) = e^{j\omega}$. Then the phase calculated using the standard two-quadrant function $\mathrm{Arg}_{II}[H(e^{j\omega})] = \operatorname{atan}(\mathrm{Im}[H(e^{j\omega})]/\mathrm{Re}[H(e^{j\omega})])$ looks like this:

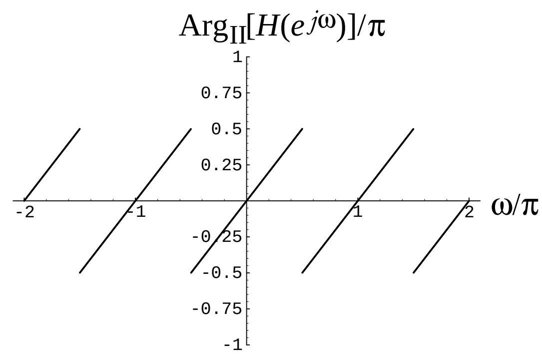


Figure 1. $H(e^{j\omega}) = e^{j\omega}$. Phase incorrect.

This anti-symmetrical function only uses two of the available quadrants. On the other hand, the four-quadrant function $\mathrm{Arg}_{IV}[H(e^{j\omega})] = \operatorname{atan}(\mathrm{Re}[H(e^{j\omega})], \mathrm{Im}[H(e^{j\omega})])$ looks like this:

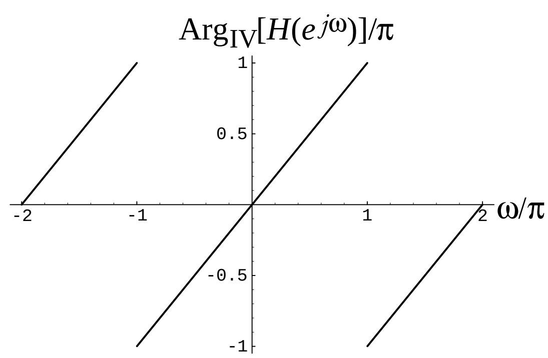


Figure 2. $H(e^{j\omega}) = e^{j\omega}$. Phase correct.

The phase function $\mathrm{Arg}_{IV}[\cdot]$ uses all four quadrants. Note that both $\mathrm{atan}(v/u)$ and $\mathrm{atan}(u,v)$ are anti-symmetrical, periodic functions; 2π-periodic in the latter (correct) case. Also note that the discontinuity in the phase is genuine, and not an artifact.

The reason that Figure 2 is correct is seen by looking at the complex function graphed directly in the u,v plane as in the Figure 3.

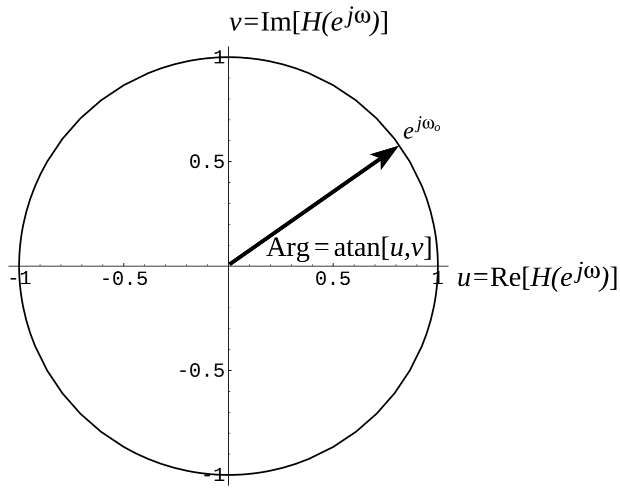


Figure 3. $H(e^{j\omega}) = e^{j\omega}$

The normalized radian frequency $\omega = 0$ corresponds to the coordinate (1, 0), while $\omega = \pm\pi$ corresponds to (-1, 0). By tracing $H(e^{j\omega})$ with ω, while measuring the angle of the vector from the origin to the corresponding point on the contour,[1] we see that $\mathrm{Arg}_{II}[\cdot]$ is clearly wrong in this case $H(e^{j\omega})=e^{j\omega}$ and, by induction, wrong in general.

[1]This graph of the complex function is circular only because $H(e^{j\omega})$ is so simple; this type of graph is usually more interesting.

Rule 2) ***Phase of any discrete signal, or phase response of any linear time-invariant discrete system is always*** 2π***–periodic in frequency*** ω.

Proof:

All discrete signals and systems can be conceived in terms of some sampled continuous signal or system. In the frequency domain, the relation between the discrete and continuous signal is [O&S,pp.83-87,ch.3.2]

$$
\begin{aligned}
X(e^{j\omega}) &\Leftrightarrow x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
X(e^{j\omega}) &= X\left(\frac{\omega}{T}\right) * \frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right) \\
&= \frac{1}{T} \sum_{k} X\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)
\end{aligned}
$$

As a consequence, notice that

$$X\left(e^{j(\omega+2\pi p)}\right) = X\left(e^{j\omega}\right) \qquad \text{; for } p \text{ an integer}$$

Going one step further, we also have that the Laplace transform of a sampled signal or system[2] is periodic in $2\pi/T$-wide strips of the s plane oriented perpendicularly to the $j\omega$ axis; viz., [M&C,ch.6]

$$
\begin{aligned}
X(e^{sT}) &= \int_{-\infty}^{\infty} \left\{ x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \right\} e^{-st}\, dt = \sum_{n=-\infty}^{\infty} x[n] e^{-sTn} \equiv X(z) \qquad ; z = e^{sT} \\
X(e^{sT}) &= X\left(e^{(s + j\frac{2\pi}{T}p)T}\right) \qquad ; s = \sigma + j\,2\pi f, \quad p \text{ an integer}
\end{aligned}
$$

The generally complex $X(\cdot)$ always has a magnitude and phase representation $|X|\, e^{j\,\mathrm{Arg}(X)}$ where $\mathrm{Arg}(X)$ is the phase response. Hence, periodicity of $|X|$ and $\mathrm{Arg}(X)$ along the $j\omega$ axis is irrefutable. This observation applies whether or not the continuous-time signal is bandlimited. ◊

[2]Note that the two-sided time-domain impulse train (the shah) has no Laplace transform.

Rule 3) ***Phase Transition***

- Zeros <u>outside</u> the unit circle cause <u>negative-going</u> transitions in phase.
- Poles <u>inside</u> the unit circle cause <u>negative-going</u> transitions in phase.
- Zeros <u>inside</u> the unit circle cause <u>positive-going</u> transitions in phase.
- Poles <u>outside</u> the unit circle cause <u>positive-going</u> transitions in phase.
- When a pole or zero is right on the unit circle there is a discontinuity in phase of π, but the direction is difficult to predict.

These bullets are summarized in Table 1 and the accompanying Figure 4.

Table 1. Phase Transition vs. Pole or Zero

	Pole	**Zero**
Inside	–	+
Outside	+	–
On	$\pm\pi$	$\pm\pi$

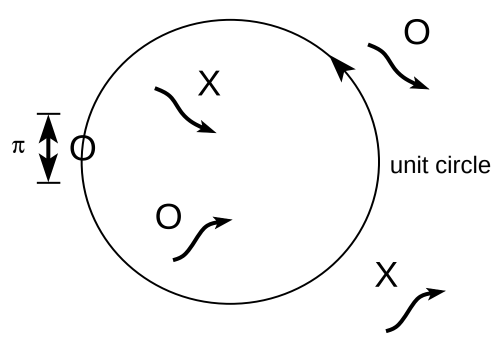


Figure 4. Phase transitions due to poles and zeros.

Vector Form of the Transfer Function

These results may be easily understood by considering the rational transform description of a signal or linear time-invariant system as a collection of vectors. [O&S,ch.5.3] The vectors are revealed when the system function is written in its rational factored form;

$$H(z) = \kappa \frac{\prod_i (z - z_i)}{\prod_i (z - p_i)}$$

for κ some constant.

Example 2

Suppose we have the transfer function $H(z) = (z - z_o) / (z - z_p)$.

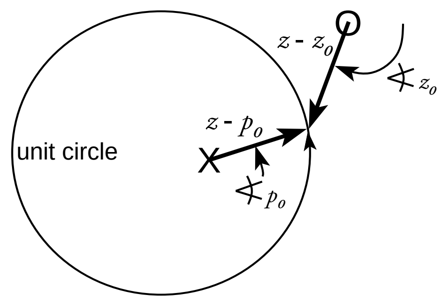


Figure 4A. Understanding phase transition in terms of vectors.

In the example shown in Figure 4A, there is a pole at p_o and a zero at z_o. The vectors $z{-}p_o$ and $z{-}z_o$ are shown for z evaluated along the unit circle. As we move around the unit circle in a counter-clockwise direction, the angle associated with the zero changes negatively at the illustrated instant while the angle associated with the pole changes positively. But because the pole vector is in the denominator of $H(z)$, the change ascribed to the pole angle is actually negative.

Phase Wrap

A discontinuity in phase of 2π within an open[3] 2π-period in frequency, is called a trigonometric *wrap* and is caused by a branch cut [Churchill] in the trigonometric function definitions. It comes about when observing the **principal value** $(-\pi < \arg[\cdot] < \pi)$ of the inverse trigonometric functions. [O&S,ch.5.3] Eliminating the discontinuities due to phase wrap has been studied.[4] [Steiglitz]

MidSummary

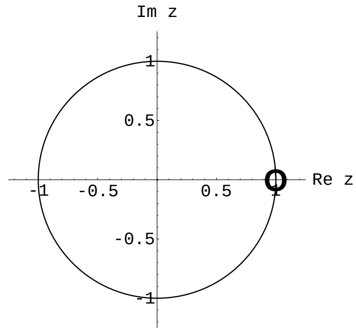
What we know thus far is that phase is 2π-periodic for discrete signals and systems, and that phase for real signals and systems must be anti-symmetrical with respect to frequency. We have also seen that phase transitions are produced by the poles and zeros of signals or systems that can be expressed in those rational terms. Now we must determine if this information is of any value to us; e.g., can we use it to sketch a phase response curve simply by viewing a given pole/zero constellation. The best approach seems to be the vector approach discussed under Rule 3. We now look at some pertinent illustrations of pole/zero constellation, magnitude, and phase.

[3] *Open* means not including the end-points of the interval 2π.

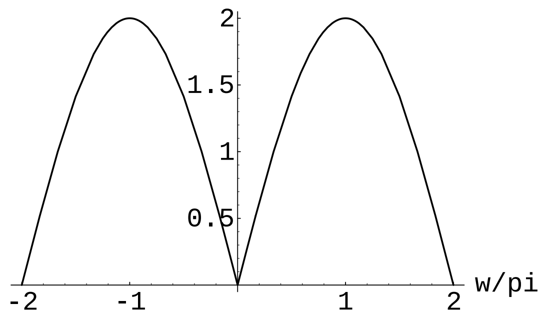
[4] The Matlab function called unwrap() is not flawless.

Table 2. Elemental Phase Response

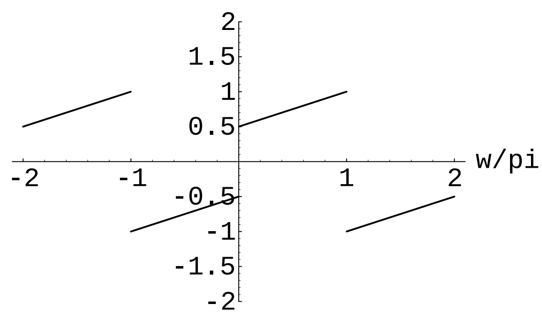
$H(z) = z - 1$



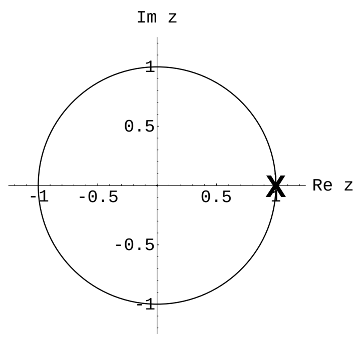
2 |Sin[ω/2]|



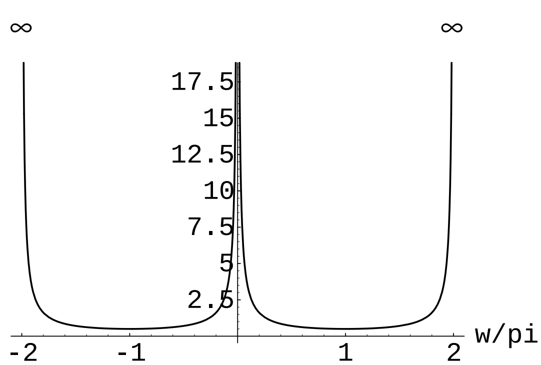
ArcTan[-2 Sin²[ω/2], Sin[ω]] / π



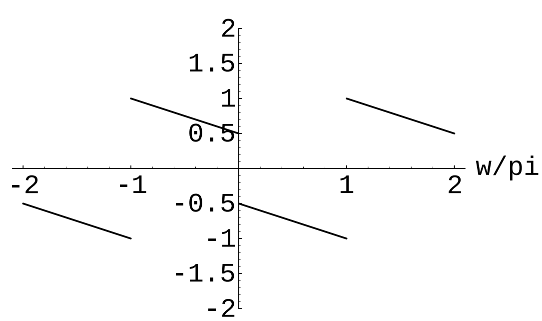
$H(z) = 1/(z - 1)$



(2 |Sin[ω/2]|)$^{-1}$

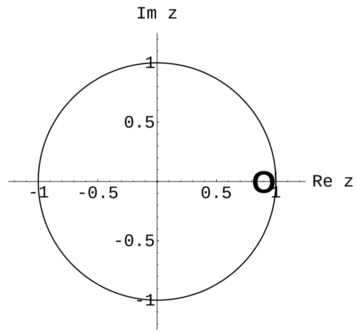


ArcTan[-2 Sin²[ω/2], -Sin[ω]] / π

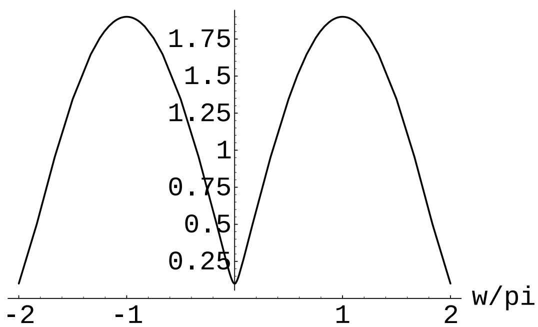


Comments: Phase is 2π-periodic on the right and left-hand sides for both illustrations.
The magnitude is the reciprocal of the previous illustration, while the phase is the negative of the previous illustration.

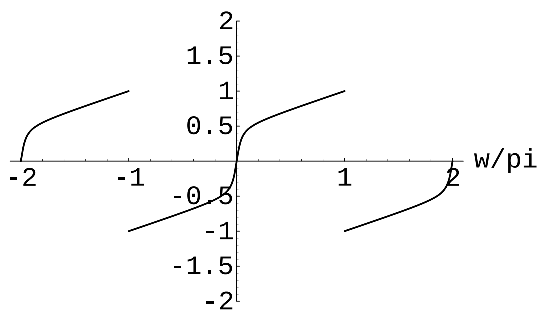
$H(z) = z - 0.9$



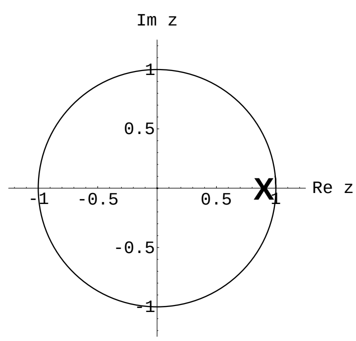
$\sqrt{(1.81 - 1.8\,\mathrm{Cos}[\omega])}$



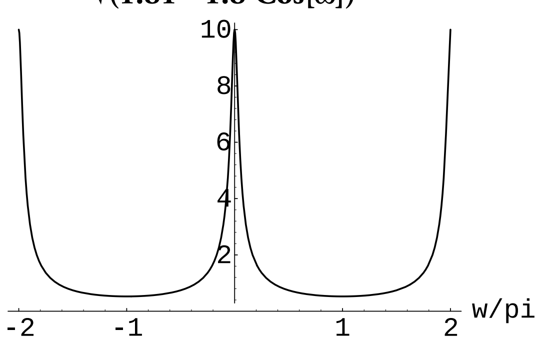
ArcTan[-0.9 + Cos[ω], Sin[ω]] / π



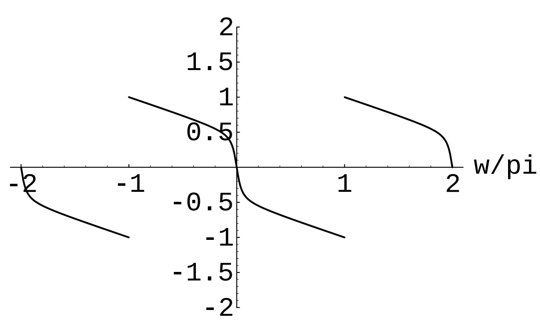
$H(z) = 1/(z - 0.9)$



$\sqrt{(1.81 - 1.8\,\mathrm{Cos}[\omega])}^{-1}$

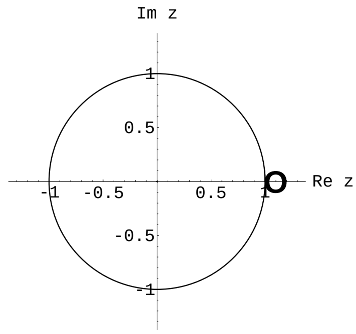


ArcTan[-0.9 + Cos[ω], -Sin[ω]] / π

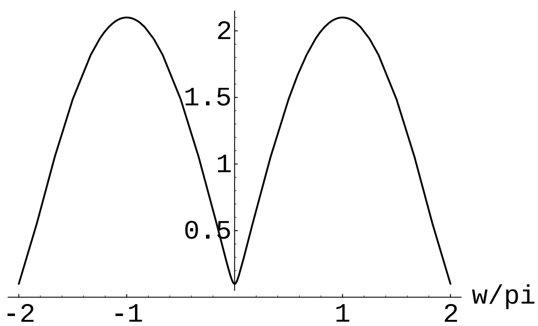


Comments: Both maximum phase.

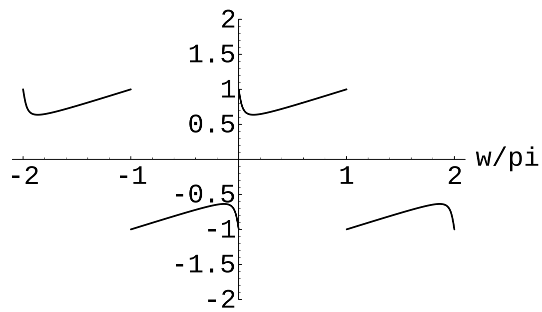
$H(z) = z - 1.1$



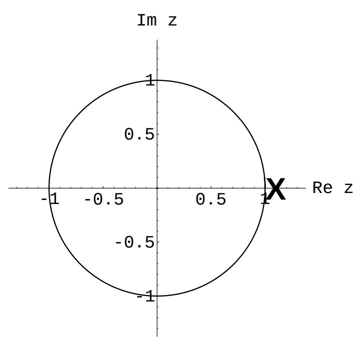
$\sqrt{(2.21 - 2.2\,\mathrm{Cos}[\omega])}$



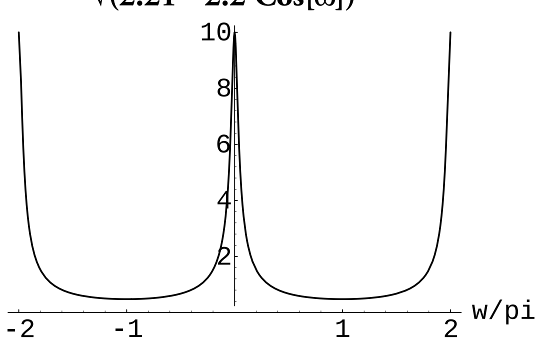
$\mathrm{ArcTan}[-1.1 + \mathrm{Cos}[\omega],\ \mathrm{Sin}[\omega]] / \pi$



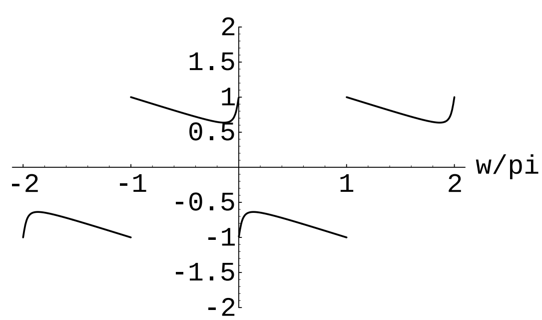
$H(z) = 1/(z - 1.1)$



$\sqrt{(2.21 - 2.2\,\mathrm{Cos}[\omega])}^{-1}$

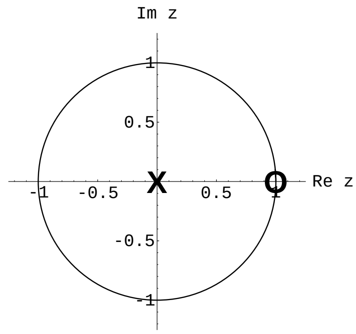


$\mathrm{ArcTan}[-1.1 + \mathrm{Cos}[\omega],\ -\mathrm{Sin}[\omega]] / \pi$

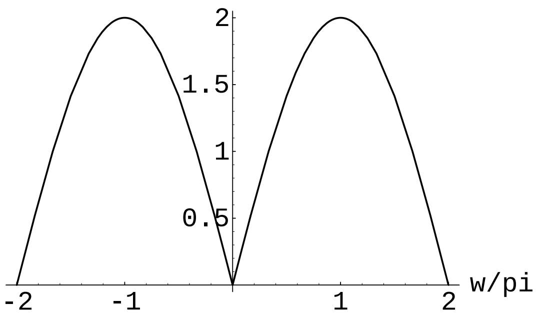


Comments: Both minimum phase.

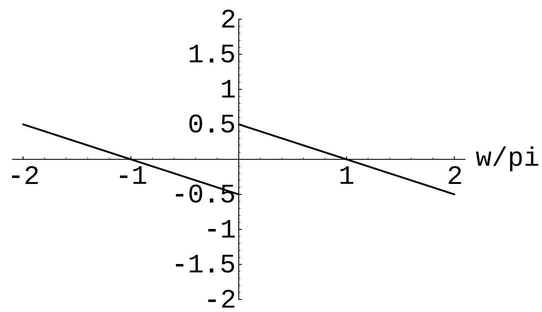
$$H(z) = 1 - z^{-1}$$



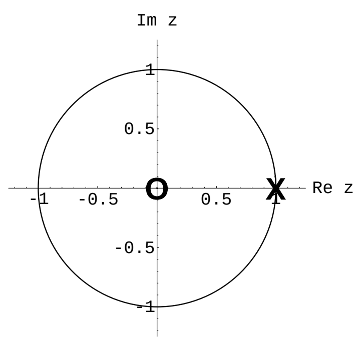
$2\,|\mathrm{Sin}[\omega/2]|$



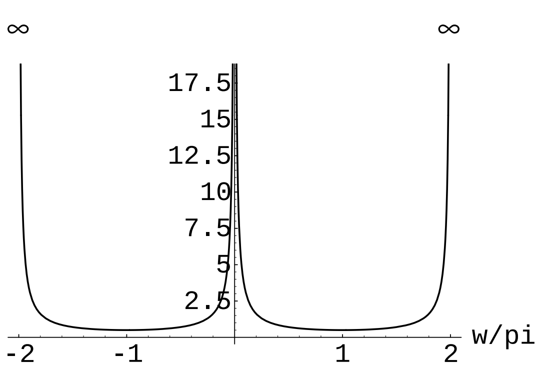
$\mathrm{ArcTan}[2\,\mathrm{Sin}^2[\omega/2],\ \mathrm{Sin}[\omega]]\ /\ \pi$



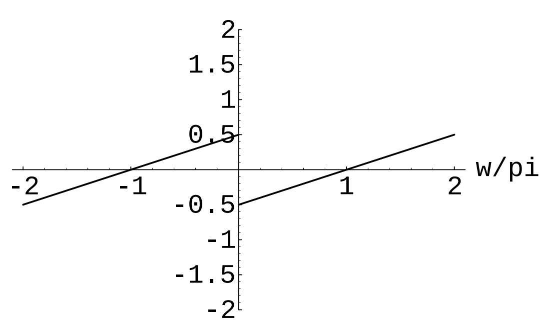
$$H(z) = 1/(1 - z^{-1})$$



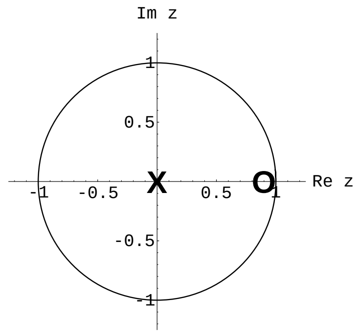
$(2\,|\mathrm{Sin}[\omega/2]|)^{-1}$



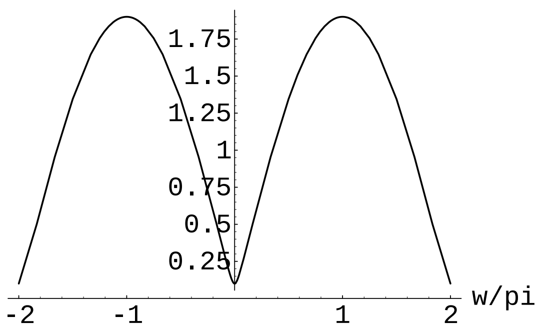
$\mathrm{ArcTan}[2\,\mathrm{Sin}^2[\omega/2],\ -\mathrm{Sin}[\omega]]\ /\ \pi$



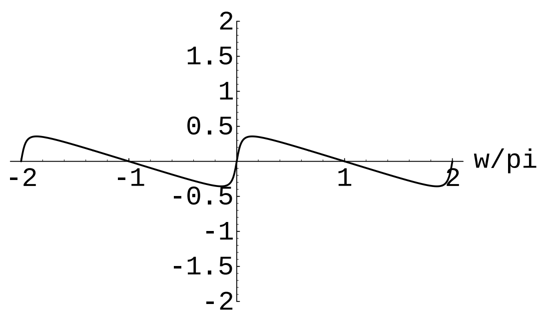
$H(z) = 1 - 0.9\, z^{-1}$



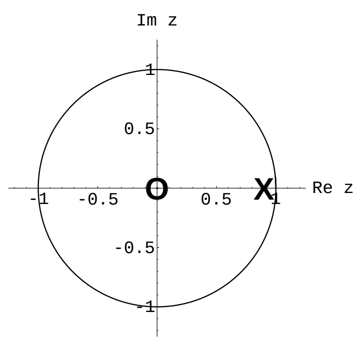
$\sqrt{(1.81 - 1.8\, \mathrm{Cos}[\omega])}$



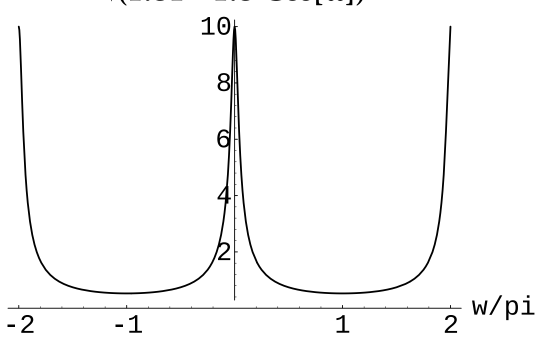
$\mathrm{ArcTan}[1 - 0.9\, \mathrm{Cos}[\omega],\ 0.9\, \mathrm{Sin}[\omega]] / \pi$



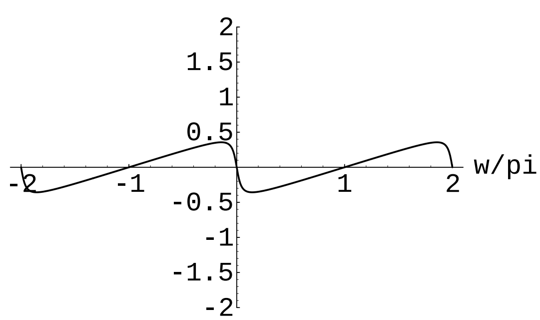
$H(z) = 1/(1 - 0.9\, z^{-1})$



$\sqrt{(1.81 - 1.8\, \mathrm{Cos}[\omega])}^{-1}$

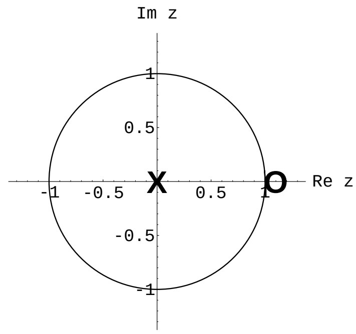


$\mathrm{ArcTan}[1 - 0.9\, \mathrm{Cos}[\omega],\ -0.9\, \mathrm{Sin}[\omega]] / \pi$

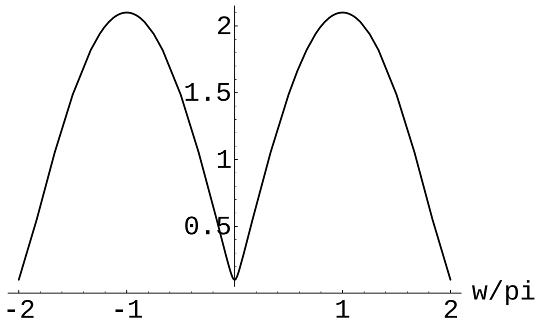


Comments: Both minimum phase.

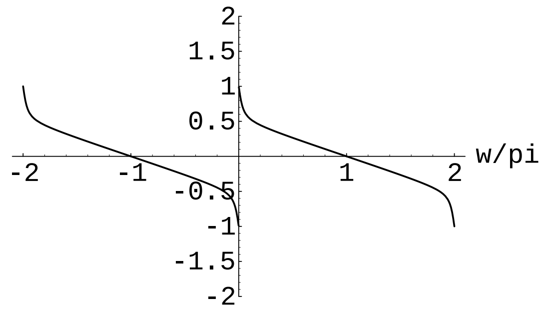
$H(z) = 1 - 1.1\, z^{-1}$



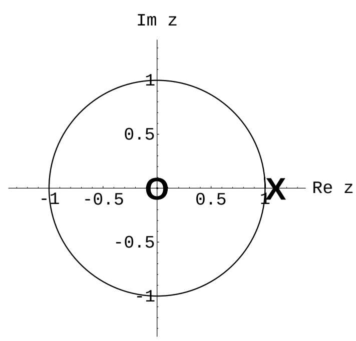
$\sqrt{(2.21 - 2.2\, \mathrm{Cos}[\omega])}$



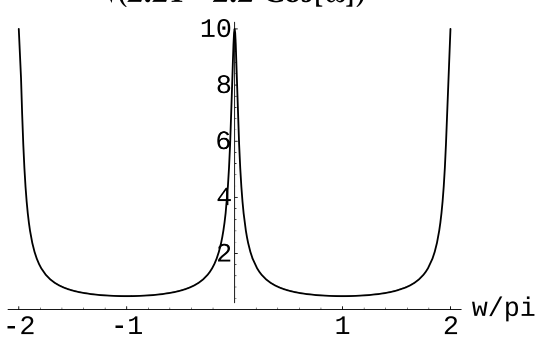
ArcTan[1 - 1.1 Cos[ω], 1.1 Sin[ω]] / π



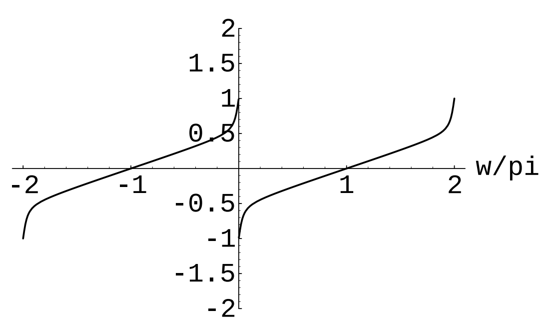
$H(z) = 1/(1 - 1.1\, z^{-1})$



$\sqrt{(2.21 - 2.2\, \mathrm{Cos}[\omega])}^{-1}$

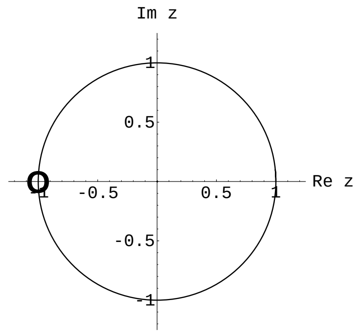


ArcTan[1 - 1.1 Cos[ω], -1.1 Sin[ω]] / π

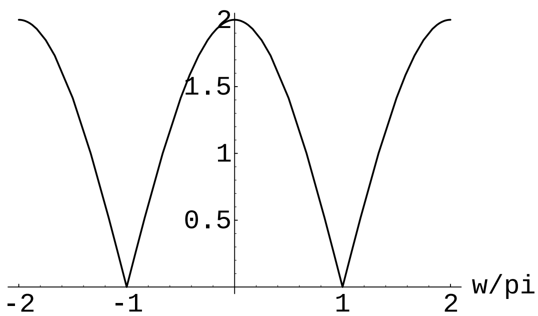


Comments: Both maximum phase.

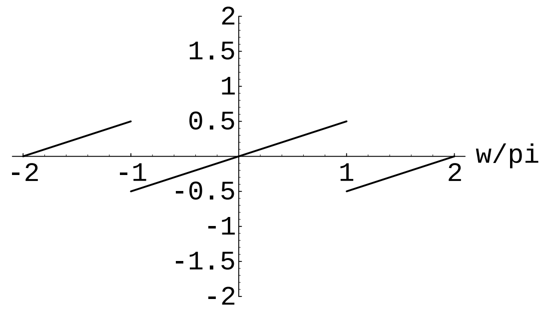
$H(z) = z + 1$



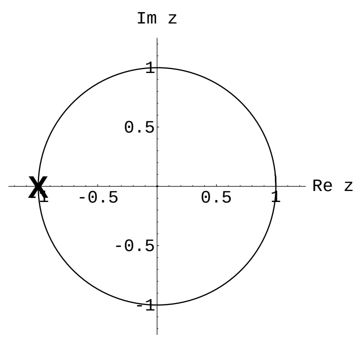
$2\,|\mathrm{Cos}[\omega/2]|$



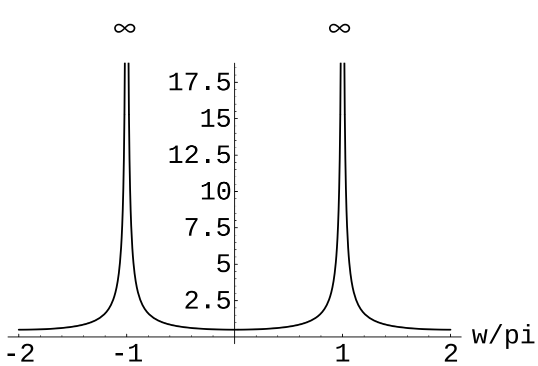
$\mathrm{ArcTan}[2\,\mathrm{Cos}^2[\omega/2],\ \mathrm{Sin}[\omega]]\ /\ \pi$



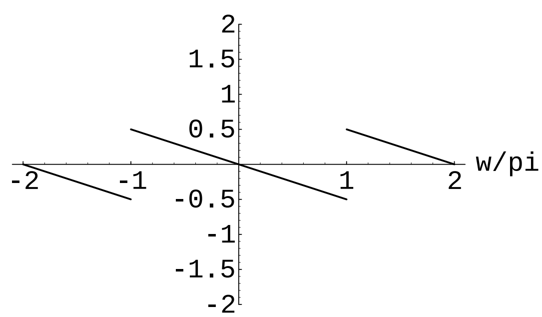
$H(z) = 1/(z + 1)$



$(2\,|\mathrm{Cos}[\omega/2]|)^{-1}$

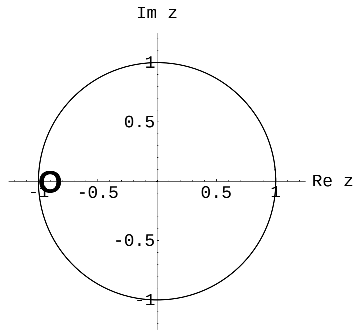


$\mathrm{ArcTan}[2\,\mathrm{Cos}^2[\omega/2],\ -\mathrm{Sin}[\omega]]\ /\ \pi$

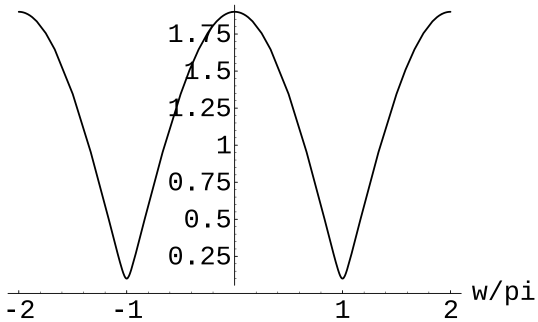


Comments: Magnitude is the reciprocal of the previous illustration, phase is the negative.

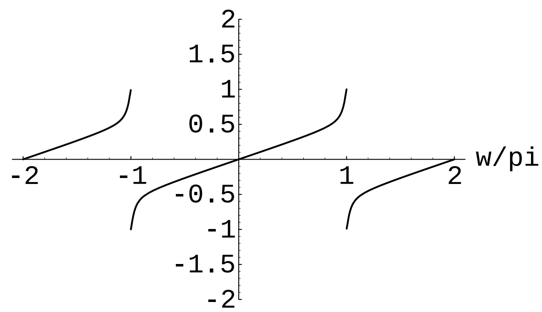
$H(z) = z + 0.9$



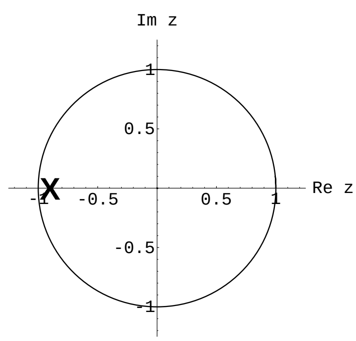
$\sqrt{(1.81 + 1.8 \, \mathrm{Cos}[\omega])}$



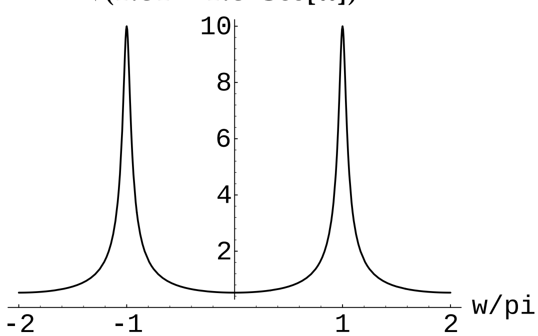
$\mathrm{ArcTan}[0.9 + \mathrm{Cos}[\omega], \ \mathrm{Sin}[\omega]] \ / \ \pi$



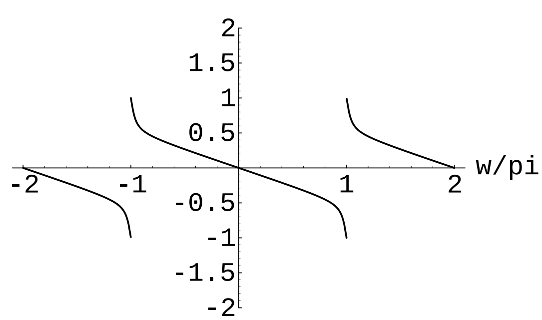
$H(z) = 1/(z + 0.9)$



$\sqrt{(1.81 + 1.8 \, \mathrm{Cos}[\omega])}^{-1}$

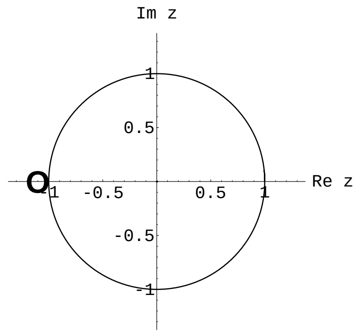


$\mathrm{ArcTan}[0.9 + \mathrm{Cos}[\omega], \ -\mathrm{Sin}[\omega]] \ / \ \pi$

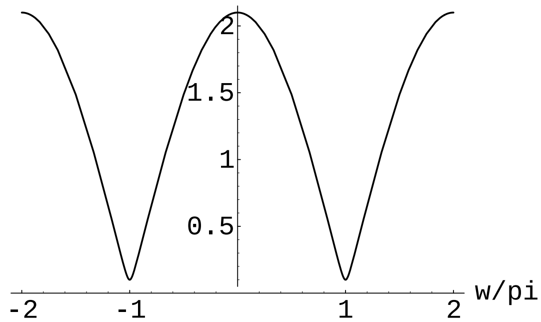


Comments: Both maximum phase.

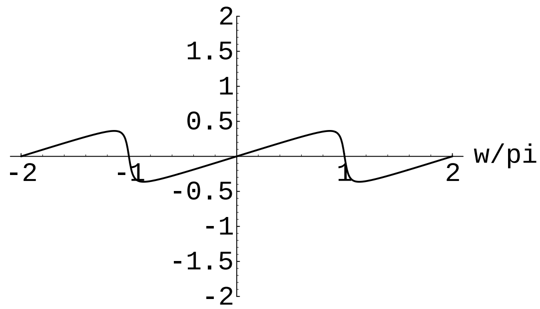
$H(z) = z + 1.1$



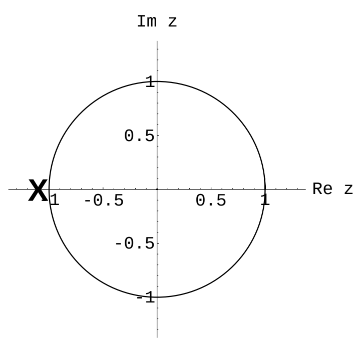
$\sqrt{(2.21 + 2.2\ \mathrm{Cos}[\omega])}$



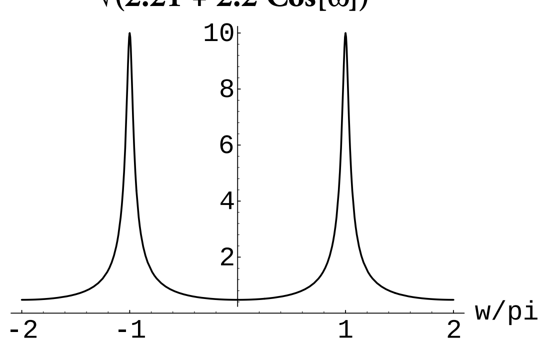
$\mathrm{ArcTan}[1.1 + \mathrm{Cos}[\omega],\ \mathrm{Sin}[\omega]] / \pi$



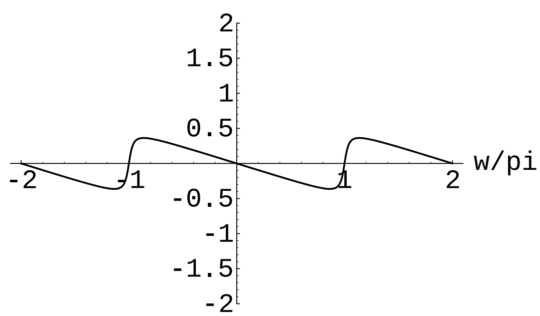
$H(z) = 1/(z + 1.1)$



$\sqrt{(2.21 + 2.2\ \mathrm{Cos}[\omega])}^{-1}$

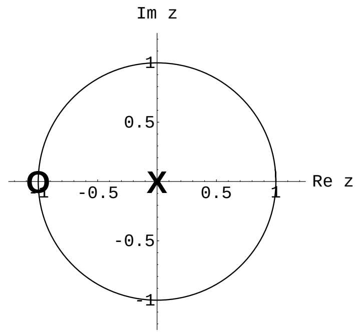


$\mathrm{ArcTan}[1.1 + \mathrm{Cos}[\omega],\ -\mathrm{Sin}[\omega]] / \pi$

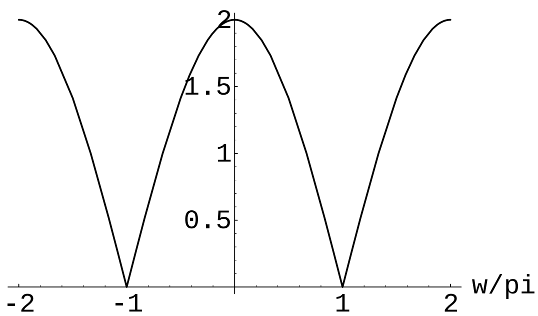


Comments: Both minimum phase.

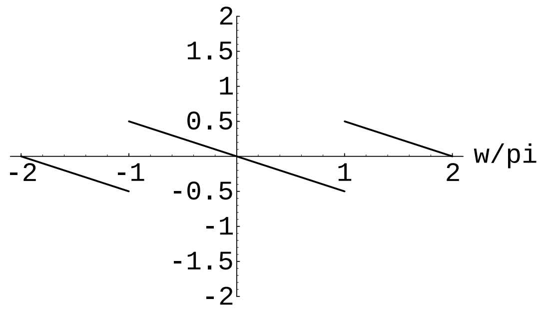
$H(z) = 1 + z^{-1}$



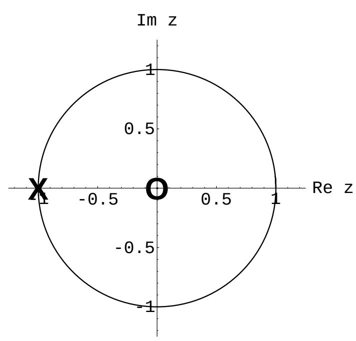
2 |Cos[ω/2]|



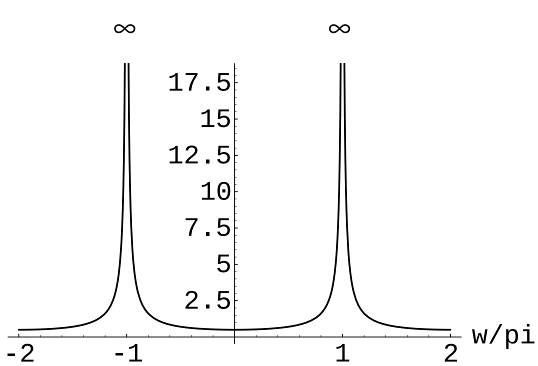
ArcTan[2 Cos^2[ω/2], -Sin[ω]] / π



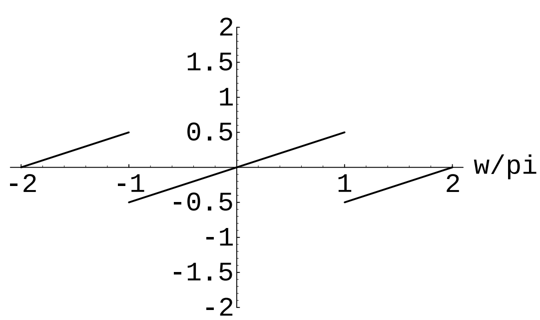
$H(z) = 1/(1 + z^{-1})$



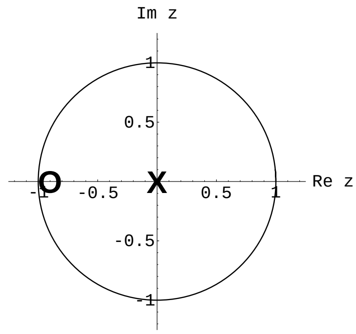
(2 |Cos[ω/2]|)$^{-1}$



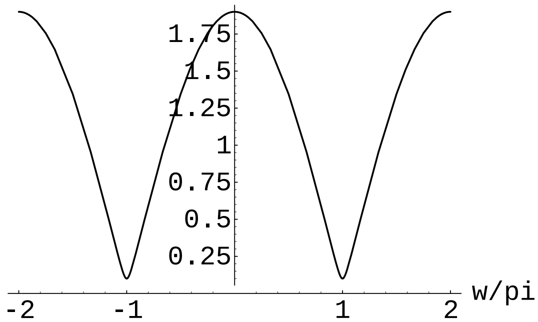
ArcTan[2 Cos^2[ω/2], Sin[ω]] / π



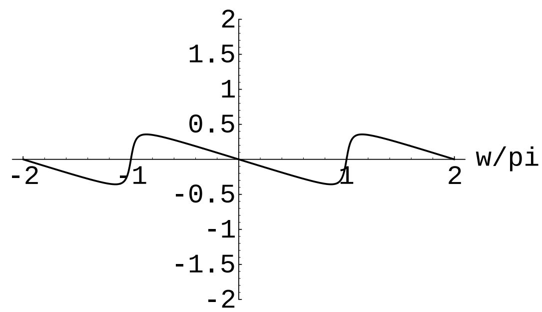
$H(z) = 1 + 0.9\,z^{-1}$



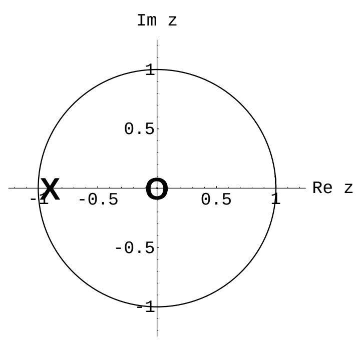
$\sqrt{(1.81 + 1.8\,\mathrm{Cos}[\omega])}$



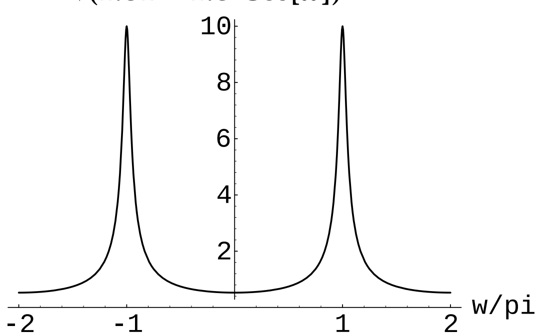
$\mathrm{ArcTan}[1 + 0.9\,\mathrm{Cos}[\omega],\ -0.9\,\mathrm{Sin}[\omega]]\,/\,\pi$



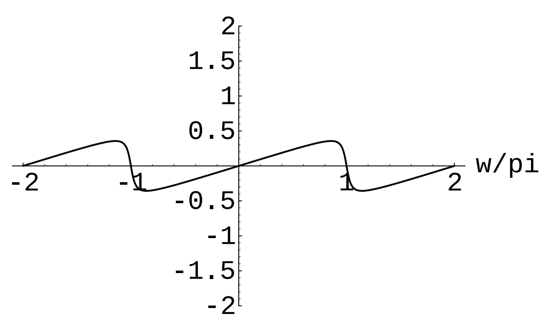
$H(z) = 1/(1 + 0.9\,z^{-1})$



$\sqrt{(1.81 + 1.8\,\mathrm{Cos}[\omega])}^{-1}$

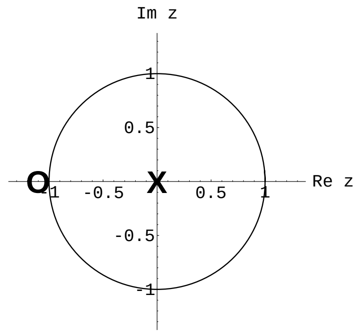


$\mathrm{ArcTan}[1 + 0.9\,\mathrm{Cos}[\omega],\ 0.9\,\mathrm{Sin}[\omega]]\,/\,\pi$

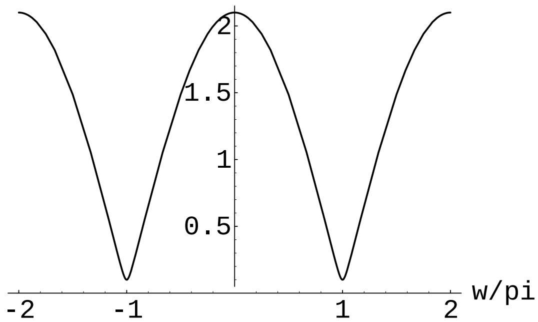


Comments: Both minimum phase.

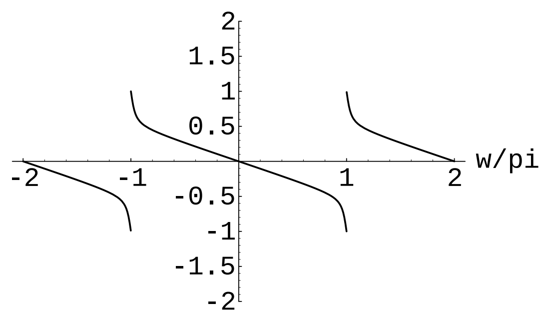
$$H(z) = 1 + 1.1\, z^{-1}$$



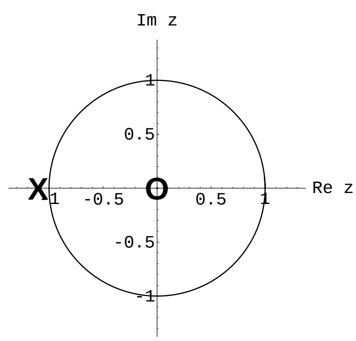
$\sqrt{(2.21 + 2.2\,\mathrm{Cos}[\omega])}$



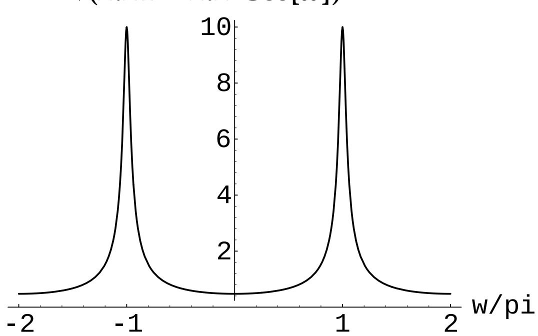
$\mathrm{ArcTan}[1 + 1.1\,\mathrm{Cos}[\omega],\ -1.1\,\mathrm{Sin}[\omega]]\,/\,\pi$



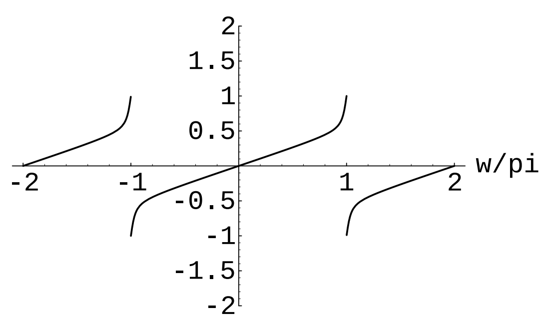
$$H(z) = 1/(1 + 1.1\, z^{-1})$$



$\sqrt{(2.21 + 2.2\,\mathrm{Cos}[\omega])}^{-1}$



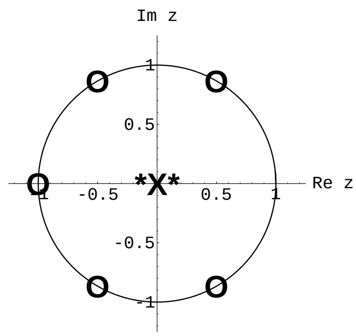
$\mathrm{ArcTan}[1 + 1.1\,\mathrm{Cos}[\omega],\ 1.1\,\mathrm{Sin}[\omega]]\,/\,\pi$



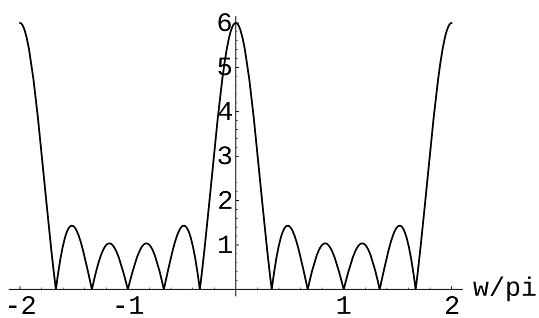
Comments: Both maximum phase.

Table 2A. Compound Phase Response

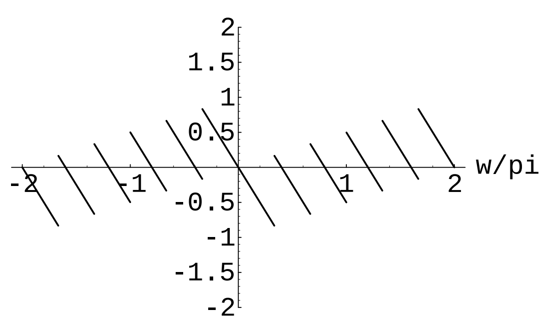
$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$



2 |Cos[ω/2] + Cos[(3 ω)/2] + Cos[(5 ω)/2]|

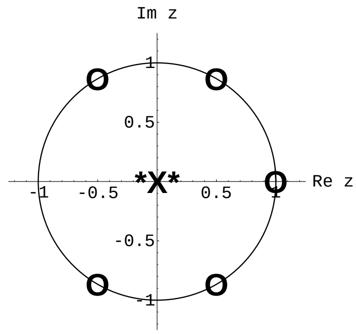


ArcTan[1 + Cos[ω] + Cos[2 ω] + Cos[3 ω] + Cos[4 ω] + Cos[5 ω],
-Sin[ω] - Sin[2 ω] - Sin[3 ω] - Sin[4 ω] - Sin[5 ω]] / π

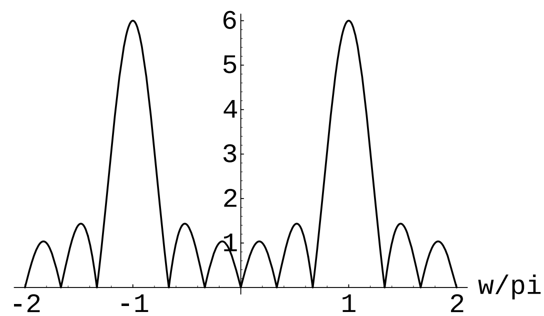


Comment: Five poles.

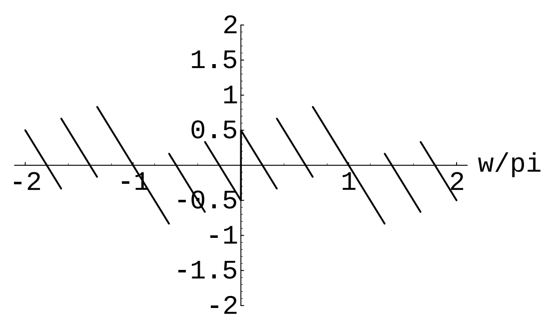
$H(z) = 1 - z^{-1} + z^{-2} - z^{-3} + z^{-4} - z^{-5}$



2 |Sin[ω/2] - Sin[(3 ω)/2] + Sin[(5 ω)/2]|

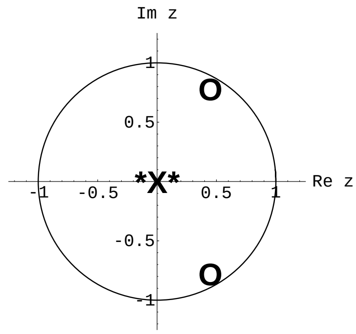


ArcTan[1 - Cos[ω] + Cos[2 ω] - Cos[3 ω] + Cos[4 ω] - Cos[5 ω],
Sin[ω] - Sin[2 ω] + Sin[3 ω] - Sin[4 ω] + Sin[5 ω]] / π

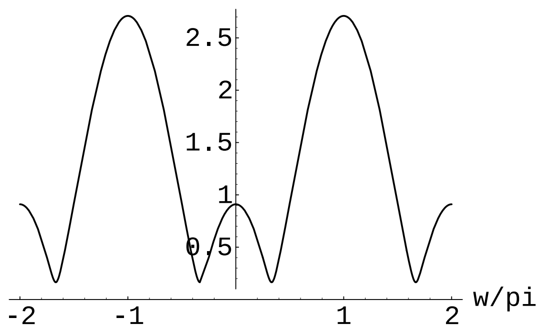


Comment: Five poles.

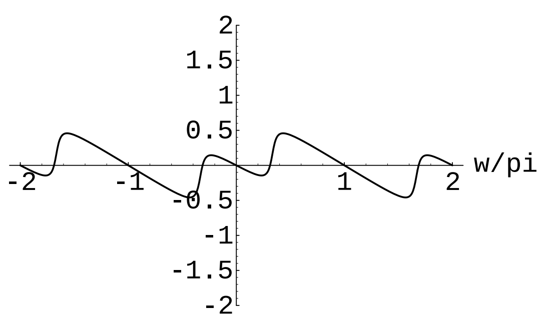
$$H(z) = (z - 0.9\ e^{j\pi/3})\ (z - 0.9\ e^{-j\pi/3})\ /\ z^2$$



$$\sqrt{[(1.81 - 0.9\ \mathrm{Cos}[\omega])^2 - 2.43\ \mathrm{Sin}^2[\omega]]}$$

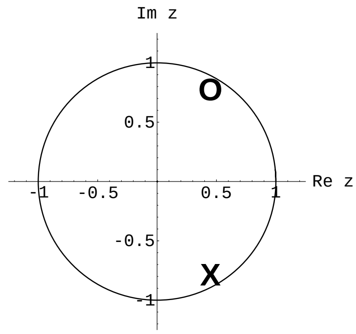


$$\mathrm{ArcTan}[1 - 0.9\ \mathrm{Cos}[\omega] + 0.81\ \mathrm{Cos}[2\ \omega],\ \ 0.9\ \mathrm{Sin}[\omega] - 0.81\ \mathrm{Sin}[2\ \omega]]\ /\ \pi$$

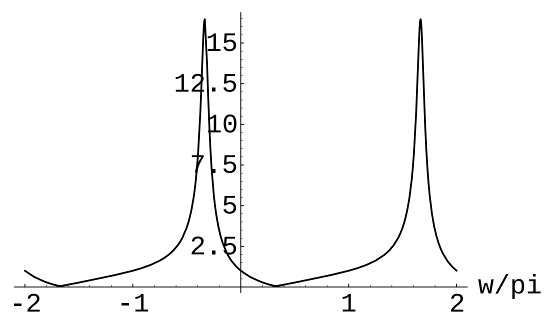


Comment: Minimum phase, two poles.

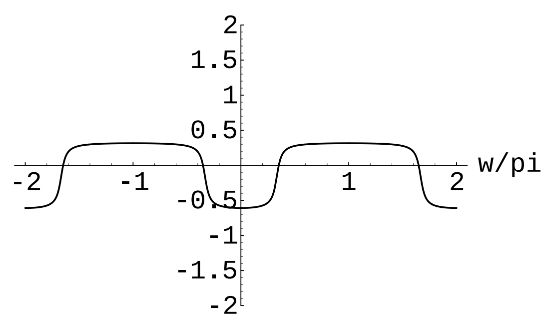
$$H(z) = (z - 0.9\ e^{j\pi/3})\ /\ (z - 0.9\ e^{-j\pi/3})$$



$$\sqrt{[(1.81 - 0.9\ \{\mathrm{Cos}[\omega] + \sqrt{3}\ \mathrm{Sin}[\omega]\})\ /}$$
$$(1.81 - 0.9\ \{\mathrm{Cos}[\omega] - \sqrt{3}\ \mathrm{Sin}[\omega]\})]$$

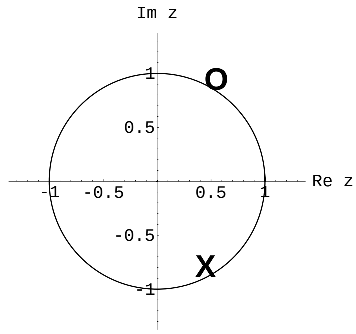


$$\mathrm{ArcTan}[0.595 - 0.9\ \mathrm{Cos}[\omega],\ \sqrt{3}(0.405 - 0.9\ \mathrm{Cos}[\omega])]\ /\ \pi$$

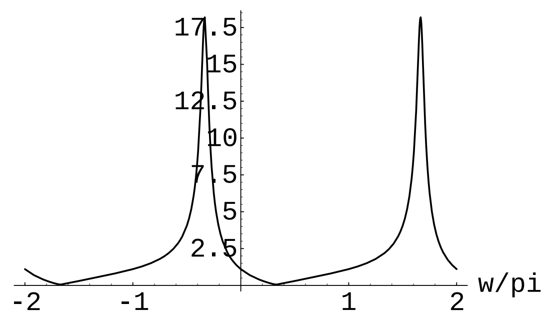


Comment: Minimum phase, complex system.

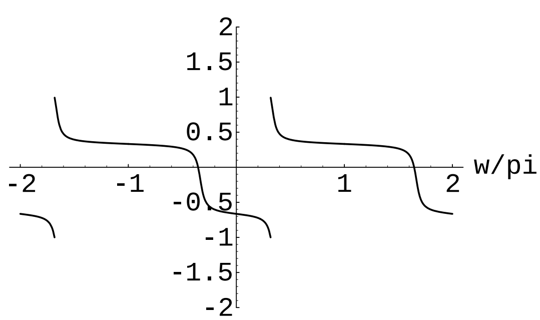
$$H(z) = (z - 1.1\ e^{j\pi/3})\ /\ (z - 0.9\ e^{-j\pi/3})$$



$$\sqrt{[(2.21 - 1.1\ \{\mathrm{Cos}[\omega] + \sqrt{3}\ \mathrm{Sin}[\omega]\})\ /\ (1.81 - 0.9\ \{\mathrm{Cos}[\omega] - \sqrt{3}\ \mathrm{Sin}[\omega]\})]}$$



$$\mathrm{ArcTan}[0.505 - \mathrm{Cos}[\omega] - 0.1\ \sqrt{3}\ \mathrm{Sin}[\omega],\ \ \sqrt{3}\ (0.495 - \mathrm{Cos}[\omega]) + 0.1\ \mathrm{Sin}[\omega]]\ /\ \pi$$



Comment: Maximum phase, complex system.

Generalized Frequency Response, Linear Phase, and Amplitude

Let's do a couple of simple examples that can be found in the elemental list of phase responses. Then we will generalize the results.

Example 3

$$H(z) = z - 1$$

We presume that this transfer function is the result of some sampling process. Thus we know a priori that it is a 2π-periodic function of frequency ω. [Rule 2] Each term of

$$H(e^{j\omega}) = e^{j\omega} - 1 \qquad ; |\omega| < \pi$$

is also 2π-periodic due to the assumption of linearity. We already know what $e^{j\omega}$ looks like from Figure 2. $H(e^{j\omega})$ can be written equivalently as

$$H(e^{j\omega}) = 2\,j\sin(\omega/2)\;e^{j\omega/2} = 2\sin(\omega/2)\;e^{j\omega/2 + j\pi/2} \qquad ; |\omega| < 2\pi \qquad (2)$$

An important point here is to recognize that the exponential term in Equ.(2) has been expanded in the frequency domain by a factor of 2 as in Figure 5; i.e., the argument of $e^{j\omega/2}$ is 4π-periodic as is $\sin(\omega/2)$. Even so, $H(e^{j\omega})$ remains 2π-periodic.

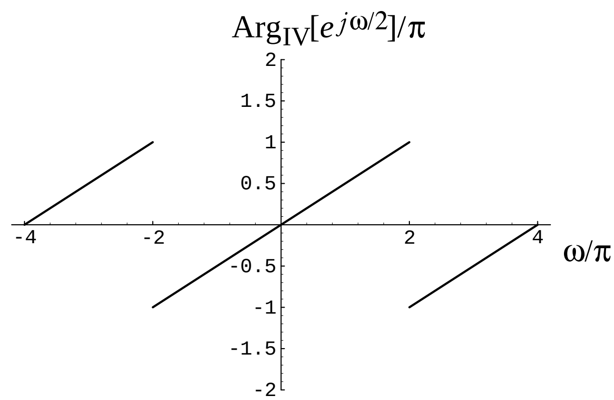


Figure 5. Phase of $e^{j\omega}$ expanded in ω by factor of 2.

If in the previous example it were erroneously assumed that the argument of $e^{j\omega/2}$ is 2π-periodic, then an incorrect magnitude and phase would ensue. That is <u>not</u> to say that the argument of the function $e^{j\omega/2}$ is always 4π-periodic. It is so in the previous example because we know its origin.

Example 4

Suppose that we are specifying an ideal allpass digital filter having an advance of precisely 1/2 sample. Then the frequency response would be stated in a magnitude and phase description like so:

$$H(e^{j\omega}) = e^{j\omega/2} \qquad ; |\omega| < \pi$$

The phase $\omega/2$ of this frequency response is 2π-periodic by definition; it must be if $H(e^{j\omega})$ is to be 2π-periodic hence representing a sequence.

Example 5

$$H(z) = 1 + z^{-1}$$

This transfer function corresponds to a sequence;

$$h[n] = \delta[n] + \delta[n-1]$$

Thus we know a priori that its discrete-time Fourier transform

$$H(e^{j\omega}) = 1 + e^{-j\omega} \qquad ; |\omega| < \pi$$

is a 2π-periodic function of frequency ω. Each term of $H(e^{j\omega})$ is also 2π-periodic due to the assumption of linearity. $H(e^{j\omega})$ can be written equivalently as

$$H(e^{j\omega}) = 2\cos(\omega/2)\, e^{-j\omega/2} \qquad ; |\omega| < 2\pi \qquad (3)$$

The important point here is to recognize that the exponential term has been expanded in the frequency domain by a factor of 2 as in Figure 6; i.e., the argument of $e^{-j\omega/2}$ is 4π-periodic as is $\cos(\omega/2)$. Even so, $H(e^{j\omega})$ remains 2π-periodic.

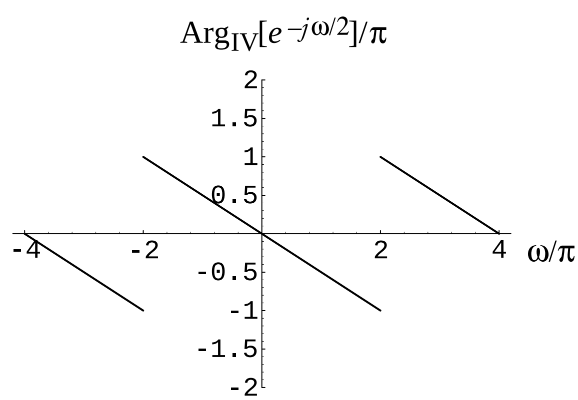


Figure 6. Phase of $e^{-j\omega}$ expanded in ω by factor of 2.

The polar form, Equ.(4) shown in Figure 7, is 2π–periodic in both magnitude and phase.

$$H(e^{j\omega}) = |2\cos(\omega/2)|\; e^{-j\omega/2} \qquad ; |\omega| < \pi \qquad (4)$$

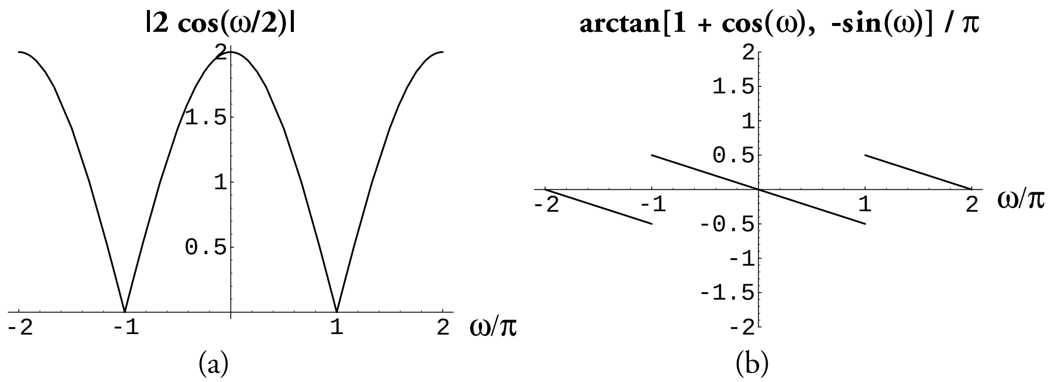


Figure 7. Magnitude (a) and phase (b) from Equ.(4) .

It is interesting that the phase of the polar form may be expressed simply

$$-\frac{\omega}{2} \equiv \arctan(1+\cos(\omega),\ -\sin(\omega))$$

This 2π-periodicity is, in fact, the conventional specification of linear phase in DSP.

Example 6

$$H(z) = 1 + z^{-2}$$

This transfer function corresponds to a sequence;

$$h[n] = \delta[n] + \delta[n\text{-}2]$$

Thus we know a priori that its discrete-time Fourier transform

$$H(e^{j\omega}) = 1 + e^{-j\omega 2} \qquad ; |\omega| < \pi$$

is a 2π-periodic function of frequency ω. Each term of $H(e^{j\omega})$ is also 2π-periodic due to the assumption of linearity. $H(e^{j\omega})$ can be written equivalently as

$$H(e^{j\omega}) = 2\cos(\omega)\; e^{-j\omega} \qquad ; |\omega| < 2\pi \qquad (3a)$$

The exponential term has once again been expanded in the frequency domain by a factor of 2; but in this case, it doesn't matter because both terms of Equ.(3a) are also periodic in 2π. To justify this last statement for the generalized linear phase, we allow phase wrap.

The concept of generalized frequency response was introduced to help explore the properties of linear phase FIR filters; [O&W] [S&K]

$$H(e^{j\omega}) = A(e^{j\omega})\, e^{-j\omega\alpha + j\beta} \qquad \text{[O\&S,Equ.(5.135),pg.255]}$$

where the *generalized amplitude* function $A(e^{j\omega})$ is real (including negative real) and not necessarily 2π-periodic, and where the *generalized linear phase* is defined[5] to be $-\alpha\omega + \beta$ and neither necessarily 2π-periodic. Traditionally one would write the polar form of a frequency response in terms of magnitude and phase; $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\,\mathrm{Arg}[H(e^{j\omega})]}$ any part of which is always 2π-periodic. The relaxation of the traditional magnitude constraint upon $A(e^{j\omega})$ occurs naturally when one considers certain symmetrical impulse responses of the form $h[n] = h^*[-n]$ (Type I) which have the property of zero phase (α=0, β=0), or anti-symmetrical responses $h[n] = -h^*[-n]$ (Type III) which have pure imaginary spectra (α=0, $\beta=\pi/2$).

Sequence symmetry is a sufficient condition for generalized linear phase in FIR filter design. If we move the point of symmetry away from n=0, then we can write a slightly more encompassing symmetry equation of the form $h[n] = h^*[2\alpha-n]$ (Type I, Type II), while for anti-symmetry $h[n] = -h^*[2\alpha-n]$ (Type III, Type IV), where α is the point of symmetry and 2α is an integer. When causality is thus introduced, α becomes nonzero for all FIR types and is typically equal to half the filter order $M/2$; just enough to make the impulse response causal. These results are summarized in Table 3. Type II and Type IV arise in the case that 2α is an odd integer; where the point of symmetry rests precisely between two samples, as in Example 3, making $A(e^{j\omega})$ and the generalized linear phase 4π-periodic.

[5]The case $\beta\neq0$ is also called *affine* linear phase. For *strictly* linear phase, β=0.

Table 3. Causal Linear Phase FIR Types

Type	I	II	III	IV
$h[n]$	sym.	sym.	anti-sym.	anti-sym.
$A(e^{j\omega})$	Sym.	Sym.	Anti-Sym.	Anti-Sym.
Periodicity	2π	4π	2π	4π
$A(e^{j\omega})$	Real	Real	Imag.	Imag.
M	even	odd	even	odd
α	$M/2$	$M/2$	$M/2$	$M/2$
β	0	0	$\pi/2$	$\pi/2$

Example 7

Let's look at Example 5.14 from [O&S,ch.5.7.3,pg.260] and try to derive an equation like Equ.(2) and Equ.(3) here. We have M=5 and sequence symmetry about M/2. The sequence transform is

$$H(e^{j\omega}) = 1 + e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + e^{-j4\omega} + e^{-j5\omega} \qquad ; |\omega| < \pi \qquad (5)$$

It is easy to verify that

$$H(e^{j\omega}) - e^{-j\omega} H(e^{j\omega}) = 1 - e^{-j6\omega} \qquad ; |\omega| < \pi \qquad (6)$$

and that each side of Equ.(6) remains 2π-periodic. From Equ.(6) we may infer that

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - e^{-j6\omega}}{1 - e^{-j\omega}} && ; |\omega| < \pi \\ &= \frac{\sin((M+1)\,\omega/2)}{\sin(\omega/2)}\, e^{-j\omega M/2} && ; |\omega| < 2\pi \end{aligned} \qquad (7)$$

As in Example 3 and Example 5, $H(e^{j\omega})$ is 2π-periodic, but the argument of $e^{-j\omega M/2}$ for M=5 is 4π-periodic due to expansion in the frequency domain (the halving). We recognize $A(e^{j\omega})$ in Equ.(7) as a digital sinc() that is also 4π-periodic because M is odd.

Phase Terminology *- unfinished*

These are the terms that we need to be comfortable with:

A) Phase Shift

To accomplish phase shift, +phi on right side spectrum and -phi on left side.

B) Phase Delay

C) Group Delay

D) maximum/minimum phase

Poles or zeros at infinity.

References

[S&K] Robert D. Strum, Donald E. Kirk, *Discrete Systems and Digital Signal Processing*, 1989, Addison-Wesley

[O&S] Alan V. Oppenheim, Ronald W. Schafer, *Discrete-Time Signal Processing*, 1989, Prentice-Hall, Inc., Englewood Cliffs, NJ 07632 USA, Internet: http://www.prenhall.com/

[O&W] Alan V. Oppenheim, Alan S. Willsky, *Signals and Systems*, second edition, 1997, Prentice-Hall, Inc., Upper Saddle River, NJ 07458 USA

[M&C] Clare D. McGillem, George R. Cooper, *Continuous & Discrete Signal* & *System Analysis,* third edition, 1991, Saunders College Publishing, Holt, Rinehart and Winston, Inc., Orlando, FL 32887 USA

[Steiglitz] K. Steiglitz, B. Dickinson, "Phase Unwrapping by Factorization", IEEE Transactions on Acoustics, Speech, and Signal Processing, vol.ASSP-30, pp.984-991, 1982

[Churchill] Ruel V. Churchill, James Ward Brown, *Complex Variables and Applications*, fifth edition, 1990, McGraw-Hill, Inc.