SCI220 – Foundations of Musical Acoustics
Cogswell Polytechnical College
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Week 9 – Class Notes

Strings I (FMA)

Chapter 7: Introduction to Vibration Recipes – The Plucked String

The characteristic frequencies of the vibrational modes are often called the *natural frequencies* of the system, and the modes themselves are referred to as the *natural modes.* The terminology comes from the fact that when a system is disturbed and left alone to pursue its natural tendencies, and free vibration is always composed of one or more of these characteristic modes.

7.1 Combination of Modes: The Two-Mass Chain

When a system is struck and left to its own devices, any possible motion it has is made up of a collection of the natural vibrational modes of the system. The initial amplitudes of these modes are determined by the manner of striking.

It is possible to start the vibration of a system purely in one or another of its characteristic modes. In order to do this, we must somehow arrange for the initial configuration of the system to agree exactly with the vibrational shape of the mode we wish to excite. For a complex motion system in which more than one component can be heard, the loudness of these components will depend on the amplitudes of the two characteristic vibrations.
Any wave shape can be created for a vibrating system by combining suitably chosen initial amplitudes of the complete set of characteristic mode shapes belonging to the system.

We can also make the following assertions:

1. A system released from an initial shape identical with any one of the natural modes of vibration keeps this shape as time goes on. Its amplitude dies away in a manner described by its own halving time.
2. A system released from an initial shape that is constructed out of a set of characteristic mode shapes will not keep its initial shape as time goes on.

7.2 Vibration Recipe of a String-like Beaded Chain

A *flexible string* is a closely beaded chain in which the connecting springs have stretchability, but because of the nature of their attachment to the beads they provide no resistance if their own to the bending of the chain. When such a chain is pulled tightly between two supports, it is the stretching elasticity of the springs that provide the forces guiding any oscillation taking place.

The connections that exist between the location of the plucking point on a string and the amplitudes of the various vibrational shapes that form the components of the initial string shape can be described by the following:

1. One cannot excite a mode by plucking the string at a point where there is a stationary point or node. Such modes will not be part of the vibrational recipe.
2. For a given plectrum force, a mode gets its strongest excitation if the string is plucked where the mode has its largest excursion.
3. For a given force, the excitation of a mode is proportional to the size of the mode's excursion at the plucking point.

Plucking anywhere in about the middle third of any string hump will produce an excitation that is more that 85% of the maximum possible excitation.

The strength of excitation of the nth of out sequence of characteristic string modes depends on the following relationship:

4. The strength of excitation (by plucking) of the nth vibrational mode of a string fixed at both ends is inversely proportional to the square of the more number. That is, $a_n = (1/n)^2$ times the other proportionality.

7.3 The Basic Recipe of a Plucked or Struck String

If we wish to remove the nth mode and its whole number multiples from a vibrational recipe, we have merely to pluck (1/n)th of the way from one end of the string. Plucking near one of these spots rather than exactly on it gives the corresponding list of modes a weak rather than zero excitation. Higher-numbered modes are very weakly excited in comparison with the lower-numbered ones due to the $1/n^2$ factor.

The vibration recipe for a string that is struck by hard, sharp-edged object at a given point along its length (rather than plucked) is almost exactly the same as the plucked case. The only difference is that the strength of excitation is given now by $a_n = 1/n$.

The amplitudes of a struck string fall less rapidly as we go up the series of mode numbers than in the case of the plucked string.

There is still considerable similarity between these two excitation methods. In both cases higher modes are progressively less strong excited than are the lower-numbered ones. However, higher modes are more strongly excited by striking than by plucking a string.

Chapter 8: Broad Hammers and Plectra, Soft Hammers, and the Stiffness of Strings

8.1 The Equivalence of Broad Plectra to Sets of Narrow Ones

In principle there is no limit to the number of plectra we could use to obtain any initial shape for a string; a sufficient number of them can be properly located and pulled aside the proper distance to produce any desired initial shape. In other words, any collection of narrow plectra can take the place of whatever real world combination of fingertips, fingernails, and sharp-edged picks.

The vibration recipe associated with any given initial shape may be deducted in a straightforward way simply by combining the recipes that would be produced by the narrow plectra acting one at a time.

We can say the following regarding multiple plectra:

1. The vibration recipe produced by several plectra acting simultaneously is found by combining the recipes belonging to each plectra acting alone.
2. In combining several recipes one must remember that they may have to be added or subtracted to take care of the fact that certain plectrum positions release a given mode so that it initially starts moving in the opposite direction to that produced by a plectrum located elsewhere.
3. Two plectra pulling in the same direction will act very much like a single plectrum pulled back twice as far, as long as we are considering modes for which the inter-plectrum distance is less than one-third of the length of the hump.
4. Two plectra pulling in the same direction will produce zero excitation of any mode for which the hump length along the string is exactly equal to the distance between the plectra (or for which the inter-plectrum distance is exactly an odd number of hump lengths). This is true regardless of where along the string the pair of plectra may be placed.
5. Modes whose hump lengths are only a little larger or smaller than the inter-plectrum distance are weakly excited since they approximately meet the criterion for zero excitation given by assertion 4.

The plectrum width W, acts on a string in very much the same way as a pair of narrow plectra separated by the distance W. The width of this plectrum directly influences the total number of string modes that is it able to excite. The mode number in which no excitation is possible is given by the following relationship:

$$n = L/W$$

For this mode number n, and neighboring modes there is very little excitation. The $1/n^2$ factor given by the displacement amplitude of plucked strings assures that for modes above n there is very little excitation. The vibrational recipes for modes whose hump lengths are more that three times the length of the string covered by the thumb and forefinger can be calculated quite accurately by simply making the following substitution:

$$n = L/3W$$

Both of these calculations do not give any details on how the modes in between these two boundary modes behave, however we can assume that the modes in this region will progressively diminish.

It must be noted that this method breaks down for modes having serial numbers above n = 40 (due to stiffness effects). Despite this limitation, on our ability to estimate recipes is useful because modes above n = 30 are weakly excited. Recall that the $1/n^2$ factor that governs the displacement of plucked strings and a factor of n that governs the transformation to the force exerted by the string on the bridge (or string support). These reduce transmissions of the initial string amplitudes by a factor more than 1/30 below the amplitudes of lower frequency components so they make a nearly inaudible contribution to the sound.

8.2 The Effect of Hammer Width on the Recipe for a Struck String

When a hammer strikes a string, the resulting vibrational recipe produced by a wide soft hammer differs from that resulting from a narrow hard hammer in ways that are similar to those found in a plucked string by a narrow or broad plectrum. However, there are two parts to what are the so-called “piano-designer's” problem.

Typically the impulse transmitted to a string during the blow of a real hammer is not distributed uniformly along parts parts of the string touched by the hammer. The central part part of the hammer is firmer than the edges, since its supported on both sides by the rest of the material, so that the momentary forces exerted on the string by the center of the hammer are larger than those exerted by the edges.

For definiteness we will use W_h to indicate the width of the hammer in the sense that within the range W_h of position along the string, the force rises to values that are at least half the magnitude of the maximum force F_{max} exerted at the central part of the struck region.

The width of W_h plays a role exactly analogous to that of the inter-plectrum distance. The nature of a struck-string vibration recipe by a broad hammer with a width W_h can be summarized as:

1. Vibrational modes for which W_h is less than one-half the length of a hump are excited in almost exactly the same way as by a narrow hammer.
2. Modes for which the humps are approximately equal in length to W_h are excited about half as strongly as the would be by a narrow hammer.
3. If W_h extends over two or more humps of a vibrational mode shape, the mode receives almost no excitation.

It is worth noticing how these statements are similar to those dealing with wide plectra. There are small numerical differences, but the overall qualitative behavior is exactly the same.

8.3 The Effect of Impact Duration on the Recipe for a Struck String

When any real hammer strikes a string, not only does the strength of its briefly exerted force vary from point to point along the region of W_h in which it strikes the string, but also the force varies in time during the duration of the impact. The magnitude of the force varies during the blow in a straightforward manner, mainly because of the progressive compression of the hammer material during the first part of the collision and its subsequent relaxation during the latter part as the hammer rebounds.

See fig 8.6 and 8.7 (p.116-117)

This sort of time variation, know as *hanning impulse,* is drawn as a short segment of a sinusoid. The letter T_h indicates the time duration of the hammer blow. T_h is the time interval over which the force is equal to or greater than half of the maximum force F_{max} that is exerted during the collision.

We can now summarize the following statements:

1. Vibrational modes for which T_h is less than one-half of the time interval P/2 are excited in almost exactly the same way as by a hypothetical hammer that strikes and rebounds instantly.
2. Modes for which the time interval P/2 is approximately equal in length to the collision time T_h are excited about half as strongly as they would be by a hammer that strikes and rebounds instantly.
3. If T_h extends over a time of one or more oscillatory periods, the mode receives almost no excitation.

8.4 The Effect of String Stiffness on the Excitation of Strings

When one pulls a real string aside by means of a knife-like plectrum, the stiffness of the string prevents it from assuming a sharp marked triangular shape.

There is a close relation between struck and plucked strings, so it is at least plausible that the presence of curvature at the plucking point will eliminate the higher modes from the vibration recipe in a way that is similar to the elimination resulting from blows by a soft or round-faced hammer.

1. The presence of string stiffness will greatly reduce the excitation of higher modes in a manner similar to that resulting from the use of wide plectra or hammers.
2. Increasing the string tension and/or reducing the string stiffness will make the vibration recipe of a real string behave progressively more like that of a flexible string.

8.5 The Upper Limits of the Vibration Recipe: A Summary

Review, it contains all of the main points of the chapter.

Appendix

The natural frequency of the mode number n is given by:

$$f_n = n(v_t/2L)$$

where v_t is the velocity of the transverse wave on the string (not the speed of sound in air). v_t depends on the following relationship:

$$v_t = \sqrt{(T/\mu)}$$

T is the tension of the string and μ is the string's mass per unit length. We can combine both formulas into a single one:

$$f_n = (n/2L)\ \sqrt{(T/\mu)}$$

The tension of a string is given by the following relationship:

$$T_{max} = (\pi/4)d^2H$$

where d is the string diameter and H is the tensile strength characteristic of the string material.

The wave impedance of s thin flexible string is given by:

$$Z_{st} = \sqrt{(T\ \mu)}$$

meaning that greater inertia and tension make for greater impedance: a thick string has larger impedance than a thin one.

Fourier theory states that the presence of a kink in the initial string shape, regardless of its location, requires a distribution of energy among the natural modes whose general trend is to fall off at 6dB per octave as the frequencies progressively go higher.