SCI220 – Foundations of Musical Acoustics
Cogswell Polytechnical College
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Week 11 – Class Notes

Keyboard Instruments (FMA)

Chapter 16: Keyboard Temperaments and Tuning Properties of the Organ, Harpsichord, and Piano

16.1 "Just" Scales: The Conventional Basis for Keyboard Strings

Musicians have been forced to devise tuning procedures for various keyboard instruments that would minimize the musical drawbacks arising from the non adjustability of their pitches. Musicians guided their tunings efforts by keeping in mind an idealized pair of so-called *just scales* based on our special frequency relationships. Meaning that all the intervals measured from tonic C, aside from the major 2nd and the major 7th, correspond exactly with the well-marked beat-free relationships previously encountered. However, we are restricted to only playing in major keys of C, G, F, D, and Bb. Otherwise, keys different than these will prove to be less acceptable to the listener.

The other major tuning compromise, equal temperament, has no perfectly tuned intervals except the octave. Some scale errors are quite large, and they are irregularly arranged relative to just tuning. Equal temperament uses an individual compromise mistuning for each particular type of interval, and these interval mistunings remain exactly the same in all playing keys. This tuning system is practical because the marked and important interval of the fifth closely approximates the special relationship that is the fifth in just tuning.

16.2 A Tuning Procedure for Setting Equal Temperament

In setting equal temperament we make use of the fact that its approximate fifths are all alike, and they are very close to but not exactly equal to their beat-free counterparts; the equally tempered frequency ratio between the two tones is 1.4983 instead of the beat-free ratio of 3/2 = 1.5. We set a series of intervals each of which is smaller than the perfect fifth by 2 cents.

Suppose the repetition rate for C4 is set to 261.6 Hz. The equal-temperament G4 is then supposed to have a fundamental frequency of 261.6 x 1.4983 = 392.0 Hz. In tuning the G4 to C4 we wish to listen for a beat repetition rate of (3 x 261.6) – (2 x 392.0) = 0.89 Hz. We must of course make sure that the G4 is tuned to the low side of the zero-beat interval.
The next step is similarly to tune the D5 relative to the newly set G4. In equal temperament, the fundamental frequency of D5 is 392.0 x 1.4983 = 587.3 Hz, so that the desired beat repetition rate between G4 and D5 is (3 x 392.0) – (2 x 587.3) = 1.34 Hz.
Our next step is to tune a beat-free (2/1) octave down from D5 to D4 before we continue by setting the A4 on the basis of the lower D.

Soon after starting the tuning series in the manner described above, it is possible to get your ears as a guide in setting intervals that sound “equally out-of-tune” without the need to count beats. However, it is a good idea to keep checking back over the intervals set earlier in order to keep some consistency in your ear tuning.

Each newly 5th can be checked against the note a 3rd below it, providing a means of watching the consistency of the tuning while it its in progress.

If the tuning has been consistently carried, the ultimate correctness will be proven by the final F-to-C interval in the complete sequence. Since the tuning started with C and worked around to F by a long series of tunings, one more tuning of the sort done so far should bring the tuning back to the original C.

Once the “circle of approximate fifths” is closed at the octave by a properly uniform set of shrunken intervals, equal temperament is achieved. Any given interval between pairs of tones should sound pretty much the same, no matter where it is tested. Finally, it is time to tune the rest of the keyboard by means of beat-free octave settings based on the tempered scale set up by C4.

16.3 A Useful Unequal Temperament: Werckmeister III

This type of temperament is sometimes used today for tuning baroque-type pipe organs, harpsichords, and other older instruments.

This tuning begins by setting a C4 reference note (291.6 Hz) and tuning E4 a perfect (zero-beat) major third above it. Following this is a series of four contracted 5ths are tuned (C to G, G to D, D to A, A to E), very much as in setting up an equally tempered scale, except that here one must shrink the tuning of each of the four intervals enough for the resulting E to be identical with E previously set. Each member of this set of mistuned 5ths has a frequency ratio of 1.4952 (5.5 cents short), so that the G4 needs to have a fundamental frequency of 1.4952 X 261.6 = 391.2 Hz, and the first beating repetition rate (3 x 261.6 – (2 x 391.3) = 2.49 Hz. In Werckmeister III tuning, the second beating rate between G and D, turns out to be 3.72 Hz.

Now that the initial set of tempered 5ths have been tuned and checked against the initial E4, it must be retuned by raising it enough to make it a perfect 5th above the A that has been determined. Then this newly set E can be used as a reference point for tuning B a beat-free 5th above it. The basic scale is completed by tuning a downward sequence of perfect 5ths, beginning from C thus: C, F, Bb, Eb, Ab, Db, Gb.

Comparing Werckmeister III and equal temperament it can be observed that in C major the major 3rd is 6 cents sharp for W-III, while equal temperament is 14 cents sharp for all keys. The fourth (in W-III) is exactly beat-free in C major, while the crucial 5ths is 5 cents flat.

16.4 Some Musical Implications: Key Mood and Modulation

In the Baroque era musicians were conscious of the changes of flavor or mooed produced when a piece of music is transposed from one key to another.

For an unequal temperament system the errors of tuning relative to the just scale differ from one key to another. Table 16.1 shows that the beats when heard in any set of intervals will change from key to key. This is enough evidence to establish the fact that music played in different keys will give a different overall impression.

When in the course of a piece of music the composer shifts (modulates) into another key, the listener is subjected to an auditory process that so to speak releases him from the musical expectations associated with the original key, while a new set of expectations is sketched out belonging to the new key.

In using equal temperament the aesthetic impact resulting from modulating from one key to another is not nearly as strong as it can be in unequal temperaments. The effect is also temporary, lasting only as long as the listener can retain some kind of aural recollection of previous events.

It is ironic that in equal temperament the increased freedom to modulate from one key to another far distant from it is purchased at the expense of a lost distinction between these keys.

16.5 Vibration Physics of Real Strings

Consider a perfectly flexible round string of length L and radius r, which is stretched between rigid supports under tension T. The density of the string material is d, and the formula for the nth natural frequency of such string is:

$$f_n = \quad n(1/Lr)\sqrt{T/d}\,\sqrt{1/4\pi}$$

For such string we see that $f_n = nf_1$; the natural frequencies form an exact harmonic series. We also notice that the frequencies are all inversely proportional to the vibrating length L, so that a 5% increase in length gives a 5% decrease in each natural frequency, and a doubling of the length lowers the resulting tone by an octave (T must remain unchanged).

On the other hand, frequencies are proportional to the square root of the tension; as a consequence it would be necessary to quadruple T in order to raise the pitch by an octave, and a 5% change in frequency would require a 10% increase in T.

For musical strings where the stiffness contribution to the frequency is very small compared with that produced by the tension, we can make use of the following formula:

$$f_n \text{ (stiff string under tension)} = nf_1 \text{ (flexible string under tension)} \times (1 + Jn^2)$$

J is a string coefficient. Notice that (Jn^2) gradually raises the successive frequencies above the nf_1 harmonic series values expected for a musically simple sound source. The value of J shows that for a given frequency the inharmonicity is reduced if one uses the longest, tautest, and slender string that meets all the other requirements which may be laid down upon a musical string.

On a piano, a harpsichord, or a guitar, we notice that the ends of the string act more like clamped bars than they do hinged ones, so that the vibrational shapes include an almost undeflected section at the string ends. We can think of the clamped-end bar as acting like a hinged-end bar having a somewhat reduced length L_c. String and hinged bar vibrational shapes are similar despite the completely different frequency relationships.

The fact that strings are coupled to a soundboard at one end means that the anchorage at that end is not absolutely rigid. If some particular natural frequency of the string happens to coincide with one of the characteristic vibration frequencies of the soundboard, the bridge is

likely to be driven into an oscillation having appreciable amplitude.
It will suffice for us to imagine that the string is anchored at one end to a massive block that is free to slide vertically on a smooth rod under the influence of a pair of springs and the oscillatory up-and-down forces exerted on it by the vibrating string.

The guitar string frequencies were very nearly in a harmonic series and the ratios are progressively widened because of the effects of string stiffness. The irregularities in the observed frequency sequence can similarly be understood to arise from the influence of the guitar body resonances that happen to be near one or another of the string mode frequencies. Returning the string to a new pitch produces little change in the stiffness effects but does rearrange the interaction of the body resonances with those of the string.
The larger the soundboard the more resonances tend to overlap, and this overlapping smoothes out and dilutes the irregularities in the frequency sequence.

16.6 Temperaments for Stringed Keyboard Instruments

A. Pitch of a Single String Sound - since the partials making up the sound of a plucked or struck string are inharmonic, we should not expect the pitch of the sound to correspond very well to that belonging to a harmonically related set of components having the same fundamental frequency (see p.315 table).
Meaning, for example, that an oboist playing exactly A440 with its set of precisely harmonic partials should not expect to feel that he is perfectly in tune with a piano string tuned to give a 440 Hz first-mode frequency component.

B. The Piano Tuner's Octave – we have defined the octave as being beat-free between all the partials of the upper musical tone and the even-numbered partials of the lower tone.
When one tunes a single string belonging to C5 so that it matches a single string to C4, both notes will be tuned until they “sound right”, but the beats are not completely removed, and there is a distinct and well-defined reduction in the aggregate amount of “tonal garbage” to be perceived when the fundamental component of C5 is set about 3 cents higher than twice the fundamental component of C4. This *piano tuner's octave* seeks to find the least obstructive beating and roughness in his octave.

C. “Perfect” Fifths and Thirds on the Piano – in the middle of the piano scale, the condition of least roughness is obtained for a musical fifth when mode 1 of the upper note of the pair is tuned about 1 cent higher than 3/2 times the mode 1 frequency of the lower note. This interval turns out to be slightly smoother than the piano tuner's octave. The third is found to produce its minimum beating condition when mode 1 of the upper note is set about 3.5 cents higher than the 5/4 ratio that relates this mode 1 of the lower member of the pair.

D. Setting Temperaments on a Piano or a Harpsichord – it is necessary in setting any of the temperaments used in keyboard music to work around a series of tempered fifths whose intervals are shrunken more or less to meet the requirements of the chosen tuning system. The particular distribution comes about from the inharmonicities of the octaves and fifths, which influence the tuning in ways that differ for the two kinds of intervals. Instruments that have design error or rusty strings might throw things off in such a way that the resulting relationships are unacceptable.

16.7 Further Musical Implications and Summary

The presence of slightly inharmonic frequency relationships between components of a piano/ harpsichord have two major musical consequences. First, the beat-free indications of the special musical relationships become more diffuse, turning into minimum-roughness relationships instead. Secondly, these special relationships no longer correspond to simple numerical values between the fundamental frequencies of the two tones.

Let's summarize the observations about the relation of piano/ harpsichord tones to each other and to tones having harmonic partials:

1. The presence of string stiffness causes the tones from a piano or harpsichord strings to be made up of slightly inharmonic partials.
2. Special musical relationships such as the octave and the fifth are found between sounds from pairs of impulsively excited strings. The beats signalize these relationships when there is a tuning error, and do not completely disappear when the exact relationship is attained.
3. Pianos, harpsichords are tuned in any desired temperament upon the basis of the minimum beating relationships.
4. The overall scale is stretched on stringed keyboard instruments as a result of tuning tones whose partials are progressively sharper than a harmonic series. Similarly, an instrument having partials that run closer together than a harmonic series can also be tuned to various temperaments. This overall stretching/compressing of a scale is the direct result of inharmonicity.
5. Because of the rapid decay of impulsive string sounds and partly because of the stretching nature of inharmonicity, equal temperament gives somewhat less discordant results in pianos and harpsichords than it does in pipe organs.
6. When the tones of an impulsively excited string are presented alternately with tones containing harmonic partials, the string sound is perceived to be a few cents sharper in pitch if the lowest partial of one tone has the same frequency as the lowest partial of the other.
7. When the sound of a repeatedly excited string is superposed on the sound of an instrument having harmonic partials, our ears apply the minimum-beat criterion in a way that requires the fundamental components of the two tones to have simple, whole-number frequency ratios.
8. Exact frequency relationships for special intervals may vary from special case to special case in a musical context when impulsively and mildly inharmonic string sounds are combined with one another or with ordinary musical sounds.

Chapter 17: Sound Production in Pianos

17.1 The Soundboard As Seen by the Strings: The Concept of Wave Impedance

The string and the soundboard meet by way of the bridge, which in turn functions acoustically as part of the soundboard. The soundboard is a two-dimensional wave-carrying medium, much in the way as a membrane.
There are some 240 other strings running over the top surface of the bridge, and these also form a kind of two-dimensional wave-carrying medium “visible” to our strings.

Acoustical theory states that any wave-carrying medium can be characterized fully by two specifications: the velocity in which waves are propagated along the medium, and the wave impedance.
The rate at which the disturbance travels from its source to the point of observation is known as the *wave velocity* or speed of sound. The wave velocity always depends on the springiness or elasticity with which one small part of the medium acts on its neighbors during a disturbance; the wave velocity depends also on the inertia of the material (amount of mass in each part). These are related in the following manner:

$$\text{wave velocity} = \sqrt{elasticity / inertia}$$

Wave impedance occurs when a disturbance is set up in some medium and travels to the boundary between in and some other medium. For a disturbance traveling from a thin wire to a thicker one or a soundboard, certain fraction of the disturbance is transmitted into the new medium and the remainder is reflected back to the original medium. The amplitudes of the reflected and transmitted waves, as well as the energy carried, depends on the ratio of impedance between the two media. If the two impedances are very different, there will be almost complete reflection with only a small fraction of energy transmitted. On the other hand, if the two media have almost equal wave impedances, there will be very little reflection and the disturbance will be almost completely transmitted across the junction. Wave impedance depends on the same two properties of the medium as the wave velocity:

$$\text{wave impedance} = \sqrt{elasticity \times inertia}$$

For a flexible string of a particular material of density d, radius r (cross sectional area = πr^2), and tension T the formulas change in the following manner:

$$\text{wave velocity} = \sqrt{\frac{T}{(\pi r^2 d)}}$$

$$\text{wave impedance} = \sqrt{T (\pi r^2 d)}$$

The tension T provides the elasticity and the product of the cross-section area provides the mass per unit length measure relevant to inertia.

For a soundboard at its driving point, the wave impedance is calculated by:

$$\text{wave impedance} = t^2\sqrt{Yw\,dw}\times(a\,numerical\ constant)$$

where *Yw* is the modulus of elasticity for the wood and *dw* is the wood density.

The wave impedance ratio between the struck string and the soundboard must be chosen to meet two conflicting requirements. First, there must be sufficient transmission of vibratory energy from the string to the soundboard that our ears are ultimately provided with a sound of satisfactory loudness. For example, if the soundboard were made out of a plate of steel 4cm thick, the wave impedance would be quite large such that we could not hear any sound from it nor the string would produce much sound into the air. On the other hand, if the disturbance is transmitted from the string into the soundboard too rapidly, the vibrations would decay very quickly and we would only hear a tuned thud.

We want the soundboard impedance to be high enough that its resonances will not play an unacceptable large role in the tuning of individual strings.

17.2 The Proportions of a Mid-Scale Piano String and the Necessity for Multiple Stringing

For a real string (with inharmonicities caused by stiffness), the vibrating length L turns out to be a fixed length that is independent of the string's thickness. The minimum inhamonicity associated with a string tightened to nearly the breaking tension depends in on the relationship between the radius r and the length L:

$$J_{min} = (r^2/L^2) \times (\text{a constant})$$

This suggests that we could use the thinnest string possible, since the length has been fixed by the frequency requirements laid down for the string. However, if we make a string too thin the transmission of vibration from the string to the soundboard is proportional to the wave impedance ratio depending on the wire radius and soundboard thickness such that:

$$\frac{string\ Z}{soundboard\ Z} = \frac{r^2}{t^2}\times(constant)$$

This relationship holds is the string tension is near the breaking point. The equation indicates that making the wire thin will mean that it will be able to drive the soundboard to only a small fraction of its own amplitude, so that only very weak sounds will be radiate into the room.

A single string on a piano having a soundboard of particular properties will sustain its tones acceptably and its tuning will be in the same behavior as discussed in the previous chapter (16). However, the loudness of the sound of the single string is inadequate and the tone lacks the liveliness normally shown in pianos having triple stringing. One way of satisfying the minimum-inharmonicty (calling a thin string) and loudness requirements is to use several strings, each with its acceptable inharmonicities and which can jointly drive the soundboard to a greater vibrational amplitude.

17.3 The Effect of Multiple Stringing on the Sound of the Piano

With any reasonable well-tuned piano, the perceived loudness at the ears (in sones) is approximately 40% higher when three strings are active than when only one is producing a sound. However, the total audibility time of the decaying tone is roughly the same whether three strings are active or only one. Also, the sound for a single string is thinner and less “interesting” that when three strings are used.

An out of tune triple stringed piano will produce a jangling sound (barroom piano sound) due to the beating of the strings, while a freshly tuned one will have little beating among the lower partials and considerable beating in the higher ones producing a pleasant shimmering sound.

A piano that is tuned such that there is an 8 cent spread amongst the three strings is considered acceptable to listeners. This spread gives a bright sound, while a larger spread up to 20 cents will give a jangling sound. Close tuning close to the near-zero beat unison will produce a “dead” sound that rapidly decays. This implies that the presence of string tunes exactly alike encourages each string to rapidly transfer its vibration more rapidly to the soundboard and then to the room.

Using the wave impedance formula we can prove this fact. For three strings:

$$\text{(tricord wave Z)} = \sqrt{3\mathrm{T}\times 3(\pi r^2 d)} = 3\sqrt{T(\pi r^2 d)}$$

The three strings acting together produce a threefold increase in the wave impedance, and thus a threefold increase in the amplitude of the bridge motion, which will ultimately produce a threefold reduction in the decay time of the vibration.

Finally, the last musical important result for the use of multiple strings comes in the form of the strings decay pattern. A blow from the hammer starts all three strings off exactly in step with each other, so that they radiate strongly to the outside world. Initially each partially quickly dies away at about the rate expected for strings in unison. However, because of their slight detuning from one another, they soon will be out of step, such that they eventually act independently. The vibration of each string decays on its own isolation at the single-string rate, and beats are produced.

As the tone of a piano dies away there is a perceived raise in pitch. While there is a clear change in tone quality, it can be explained by the fact that the lower-frequency partials become unimportant and then inaudible more quickly than do the higher partials. The amplitude of the lowest partial generally falls away more quickly than higher partials because of the slow beating rate between the strings for this component keeps their vibrations in step for a longer time, during which they suffer the accelerated decay characteristic of the cooperative effect. Because of string inharmonicity, the higher partials heard by themselves imply a higher pitch than that which our ears assign when they base their “calculation” on the lower partials.

17.4 The Action of Piano Hammers

The duration of contact in a hammer blow exerts a considerable influence on the number of characteristic modes that are excited. Modes having frequencies high enough that one or more of their oscillations take place during the contact time are, as a result, only weakly

excited. On a piano, the time of contact is only partly influenced by the softness of the hammer felt; the predominant influence arises from the way the string itself pushes back against the hammer.
The natural frequency of the system is determined by:

$$f_H = \left(\frac{1}{2\pi}\right)\sqrt{\frac{T}{MH}}$$

where T is the string tension, H is the short string segment created by the hammer blow, and M being the mass of the hammer.

The time of contact is determined by:

$$T_c = (\ 1/2f_H\ ) = \pi\sqrt{\frac{MH}{3T}}$$

Increasing either the hammer mass M or the striking distance H will lengthen the time of contact and thus reduce the number of higher partials excited in the tone.

The vibrations of the string segment H form a harmonic series whose frequencies are L/H times as high as the corresponding modes of the complete, full-length strings. During the course of the blow the new high-frequencies oscillations of the hammer and of the short part of the string are given to the complete strings in addition to the more familiar components of the vibration recipe. Thus, these extra components fill in the otherwise expected gaps in the vibration recipe when the hammer strikes at nodal points at various modes. A real piano hammer blow restores the missing components and as a result no modes are ever missing from the vibrational recipe of a piano tone.

17.5 Scaling the Strings of a Piano

The need for an acceptable tone implies not only a tolerably low value for the string inharmonicity factor J, but also a properly proportioned relationship among the hammers mass, breadth, and softness, the string tension, and the point at which the hammer strikes the strings. To get sufficient power with an adequately long decay time one must in addition arrange to get a correct ratio between the wave impedances of the string and the bridge.

The piano maker in practice avoids a great deal of the inhamonicity problem by using slender steel string (to support the tension) which is wound with one or more layers of copper wire, so as to raise the mass per unit length without adding much stiffness.
The change of tone one hears in going between wound and non-wound strings is relatively small, the inharmonicity increase due to greater stiffness of wound strings is offset by the increase in tension. Wound strings are mounted on their own separate bridge, which is designed to be stiffer and more massive. As a result, the wave impedance of the soundboard to which it is glued look considerably larger to its strings than that of the lighter and more flexible structure seen by the plain wire strings.

17.6 The Sound of a Piano

When the soundboard is excited by the string where it runs over the bridge, each frequency component excites several characteristic vibrations of the soundboard in a manner exactly similar to how a small loudspeaker excites the characteristic modes in a room.

*see statements on p.347-348