BENADE 1967 31 pp

An informal preliminary report on the acoustics of brass instrument bores (chiefly on wave equations in theory and experiment) (with E. Jansson), Case Institute of Technology, Physics Department Technical Report (April 1967).** [31 pp.]

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# An Informal Preliminary Report on the Acoustics of Brass Instrument Bores (Chiefly on the Wave Equations in Theory and Experiment)

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April 1967 [but *see* endnote]

## I. Introduction

In this account we will set down briefly and directly some of the basic physical and musical properties of the air columns that are useful in the construction of brass wind instruments. It is not our present intention to give details of various calculations, nor to describe the related experimental work, except in cases where the clarity of the discussion is helped by the presence of examples. In most cases, different aspects of the discussion will be perfectly familiar to one or another of our readers, though perhaps in a context different from the present one. In the interest of brevity, then, a certain fraction of the scholarly apparatus that normally appears in a technical report will be missing here.

It is always a matter of considerable difficulty in discussions of musical acoustics to find a form of language that is intelligible both to scientists and to musicians, or at least to indicate when one's vocabulary must be shifted from one domain to the other. It is also difficult to avoid assuming too much or too little about the detailed knowledge possessed by the reader, whether in the field of applied music, or in things mathematical or scientific. Because this is an informal report for discussion among those of us interested in the acoustics of brasses, we shall not worry too much about these problems of exposition. Consistency will frankly be missing with regard to the level of sophistication

One further matter should be mentioned at the beginning. In the interests of getting some things said briefly we shall not attempt to provide detailed references to the work of others. When references are made to papers or books and to suggestions or comments made by friends, we shall in general try to give credit for major sources of the ideas, but the absence of such credit should *not* be taken as a claim to originality.

## II. Formulation of the Problem

### II-A. The Job To Be Done by the Air Column

Let us start out by listing the things that have to be done by the air column in a brass instrument from the point of view of its musical uses, in preparation for the listing of physical constraints on the system. These physical constraints generate the mathematical boundary conditions that are used to select out the useful solutions of the wave equation for a given horn shape. Each of these useful solutions represents a vibrational mode whose frequency forms the ancestor from which (in spite of all the various complications of a real instrument) the *playing* frequency descends. For this reason it is interesting and important to find out what the normal-mode frequencies are. Once we learn what these are for a given but so far arbitrary shape of horn, we can proceed to select horn families, each of whose sets of natural frequencies are internally related in musically useful ways. After this it begins to become possible to worry about the possibilities of getting musical scales by the use of valves, the effects of horn shape on the tone, the effects of different sorts of discontinuities in the bore, the effect of various mouthpiece shapes, etc., etc.

As a start in this program, let us assume that the musician requires the natural frequencies of his instrument to belong to the harmonic series. In the most general sense, one could imagine a situation where, out of the complete set of the natural frequencies $\omega_n$ (with $n = 1, 2, 3...$), it is possible to select a subset whose frequency ratios are all the desired sequence of 1, 2, 3 times the lowest of these frequencies. The player would then ignore the in-between members of the complete set and the extra ones beyond its limits, or else use them in various ways as shortcuts for trills, etc. As a matter of tradition, however, is very nearly true that instruments are built so that the frequency in the $n$'th mode is $(n/2)$ times that of the second mode, so that all of the natural modes are used. We shall assume temporarily however that all the modes are harmonically related, until it is time to get closer to the practicalities of the musician's world. Let us say us all this again briefly, for emphasis.

Until further notice, we shall assume that the natural frequencies of a musically useful horn form a harmonic series. We shall, however, keep ourselves aware that in truth this assumption is an oversimplification in several important ways.

## II-B. The Oscillator Physics That Determines the Boundary Conditions

The next thing to look into is the matter of the boundary conditions as they are imposed on the mathematics by the physics of vibration in *blown* horns. At one end of the horn is a player who supplies compressed air under reasonably high, but steady, pressure to a valve system constructed out of a pair of lips and a mouthpiece. At the other end of the instrument is the whole outer space of air into which the music is to be sent. It is a matter of brief calculation to estimate the order of magnitude of the oscillatory sound pressure inside a musical instrument from that measured outside its bell, and to use this to get an idea of the magnitude of the pressure fluctuations inside the mouthpiece. One finds that there is reasonable agreement between the amplitude of the pressure fluctuations in the mouthpiece and the magnitude of the steady pressure maintained by the player's lungs. Experiments on this go back at least to the 19th-century work of Blaikley in England, and Bouasse describes some of his observations in the 1920s. We have made some measurements recently and Arend Bouhuys has published work on this matter in *J. Acoust. Soc. Am.* For present purposes we need only to notice that the steady blowing pressure is several orders of magnitude greater than the sound pressure at the open bell.

Considerations of the sort implied in the preceding paragraph, taken with observations of the way in which the player's lips vibrate, fit together to imply the idea that the lips act essentially as a pressure-operated valve that is forced open near the peak of the pressure variation inside the mouthpiece so that the pressure can be replenished by the player's forces. All this is described fairly completely in a qualitative manner on pages 151 through 163 of *Horns, Strings and Harmony*. Now, it is a familiar fact of acoustical theory that if an air column is forcibly driven at different frequencies by external means, one finds that the pressure amplitude of the disturbance is very large at the end of the column at some frequencies, and very small at other ones. It is not too hard to deduce from this that a horn coupled to a pressure-operated valve will best be able to maintain itself as an oscillator if it runs at a frequency for which the actuating force (pressure amplitude) within the horn is as large as possible, so that the valve is opened and closed most vigorously. These frequencies are ones at which the acoustician says that the input impedance of the horn is a maximum.

We are now in a position to set up mathematical requirements on the possible vibrations in our air column. The vibrations must be of such a type that the pressure variation due to the oscillation is as large as possible at the mouthpiece end, in the interests of maintaining an oscillation. At the bell end, we require that the vibration be of a sort that has minimum pressure amplitude, because of the manifest impossibility of pumping-up the whole universe during each positive swing of the vibration, or of evacuating it on the negative swing.

In a nutshell then, we are going to look for wave solutions for our equations such that they give large pressure excursions in the mouthpiece end, and small ones at the bell end. One parenthetical remark is in order here. What has just been written about the previous fluctuations at the open end is consistent with traditional statements. Notice however that we have deduced the maximum-pressure requirement at the other end from considerations of *maintaining* vibrations rather than making a traditional bow toward calling the mouthpiece a "closed end." It is not closed, neither is it open, but is rather an active valve device. It is a mathematical accident that the requirements on the standing wave laid down via the requirements of the lip-valve mechanism and via the closed end are identical at this point in our analysis. The accident does not persist very much farther, however, and we must not tie our hands by assuming that it does. These matters are discussed in a forthcoming report that is being published by the New York Academy of Sciences, in a monograph entitled "Sound Production in Man."

## II-C. The Basic Mathematical Problem

A familiar approximation to the differential equation for waves in a pipe of varying cross section is as given in Webster's equation (*see* P. M. Morse, *Vibration and Sound*, 2d ed.,

p. 268). A very detailed account of the properties of this equation and a bibliography of earlier work has been recently written by Edward Eisner, Complete solutions of the "Webster" horn equation, *J. Acoust. Soc. Am.* 52:138 (1972):

$$\frac{1}{S}\frac{\partial}{\partial x}\left(S\frac{\partial P}{\partial x}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

Here $S$ is the area of a constant-phase surface within the horn, $p$ is the sound pressure, and $c$ is the free-space velocity of sound.

There is a notation for this equation that has been known for many years (*see* for example Vincent Salmon's classic papers in the 1946 *J. Acoust. Soc. Am.*), but which was first pointed out to me independently by Robert Pyle. This altered notation has the double virtue of extracting from the problem the dependence of $p$ on $x$ as it is determined by the "obvious" requirement that energy be conserved in a propagating wave, and also of giving the equation a mathematical form that is similar to Schroedinger's equation, with all the physical and mathematical implications of this fact being made directly available to the acoustician.

We assume that the energy flow in the horn is constant, that is,

$$p^2(x)\,S(x) = \text{constant} \tag{2a}$$

It is convenient to define a radius $r_0$ now, such that the area of the constant-phase surface is given by

$$\pi r_0^2(x) = S(x) \tag{2b}$$

If the wave disturbances in the horn are represented by a Fourier series in time, each spectral component $p(x,t;\omega)$ can be written

$$p(x,t;\omega) = [\psi(x)/r_0(x)]e^{i\omega t} \tag{3}$$

Here the function $\psi$ may be looked at as being the "reduced wave function" from which the energy-conservation dependence has been removed, leaving everything else yet to be found from the wave equation and the boundary conditions. If we substitute the definitions from Eqs. (2) and (3) into Eq. (1), the wave equation for Webster's equation takes on the form

$$\frac{\partial^2\psi}{\partial x^2} + [k^2 - U(x)]\psi = 0 \tag{4}$$

The notation is standard:

$$k=\omega/c \tag{5}$$

$$U(x) = (1/r_0)(d^2r_0/dx^2) \tag{6}$$

Aside from the boundary conditions, the complete information about the nature of the horn is contained within the function $U(x)$, which we call the "horn function." Formally it plays the same dominant role in horn acoustics that the potential energy function plays in quantum mechanics.

So far we have not restricted ourselves to any special form for the wave fronts. It has been traditional to assume that these are plane. There is a difficulty, however, with the assumption of plane waves within the bell, which becomes quite troublesome when one tries to fit a plane wave from inside the bell to the manifestly nonflat wave that spreads out from the open end. Thus we see that the traditional approach is perhaps reasonable if one is interested in *connecting* a pair of pipes of differing cross section by means of a gentle taper, but it may not be very well adapted to the analysis of radiating structures having rapid flare.

Conceptually the plane-wave approach is based on the idea that the tapered horn is a basically cylindrical pipe of varying area. Another viewpoint might be that a flaring horn is the generalization of a *conical* pipe of varying taper. This idea seems promising when one looks at the bells of brass musical instruments. Let us take a moment to see the mathematical implications of this remark. In fairly cylindrical pipes, one has the $x$-dependence more or less sinusoidal, while the radial dependence is one or another of the Bessel functions, $J_0(\mu r)$ in particular, if there is axial symmetry to the wave. The traditional treatment of tubes and horns assumes $\mu = 0$, which is the plane-wave approximation in uniform tubes. In conical pipes of uniform angle, an axially symmetrical wave depends on the Legendre polynomials $P_\ell(\cos \mu\theta)$ for the variation of the pressure an *angular* distance $\theta$ away from the horn axis. The familiar spherical-wave theory for cones assumes that once again $\mu = 0$.

We have looked into the problem of re-representing Webster's equation in an adaptation of spherical coordinates, instead of an adaptation of cylindrical coordinates, and the results are promising. We will not deal completely with the implications of this alteration at the present time, and will only refer at the end to some experimental evidence that a new viewpoint will be required in the near future. As a first step however in improving the accuracy of the Webster equation, we will use it unchanged, except for a redefinition of the wave-front area $S(x)$ to make it into the area of a locally defined spherical cap. This cap is to be constructed normal to the central axis of the horn, and normal to the generating lines of the horn's bounding surface (*see* Fig. 1).

The idea of using spherical wave-fronts in Webster's equation came up some years ago as an implication of a letter by E. S. Weibel, "On Webster's Horn Equation," published in 1955 in *J. Acoust. Soc. Am.* He pointed out on the basis of a variational calculation that a good approximation to the wave-front shape in a horn would be the shape of the steady-flow equipotentials belonging to the horn. It is not hard to sketch these in, for a musical horn, to show that they are very similar to spheres. Let us see how the initial validity of this approximation can be established: Helmholtz's equation for the velocity-potential $\phi$ of an acoustic wave is

$$\nabla^2\phi = (1/c^2)(\partial^2\phi/\partial t^2) \tag{7}$$

The low-frequency limit of this equation is identical with that governing the steady-state electrical potential in a conducting medium:

$$\nabla^2 V = 0 \tag{8}$$

If we arrange an electrolytic tank experiment to have the same boundary conditions for the electrical current, as we have in the horn for the motion of air, then the easily observable electrical potential distribution may be used to define the equivalent distribution of velocity potential for steady air flow through the horn. (*See* Jeans, *Mathematical Theory of Electricity and Magnetism*, and Lamb, *Hydrodynamics*, for discussions of the two parts of this problem.) The connection between the pressure wave fronts that are of interest to us and the measured electrical potential distribution must be made in a slightly roundabout fashion, using the fact that (grad $p$) gives the flow of air while (grad $V$) gives the flow of electrical current (the usual relation of pressure to the time-derivative of the velocity potential is not available to us in the static case!).

Figures 2 and 3 show the results of field-plot experiments on a wedge-shaped body of water whose thin edge represents the horn axis and whose thick edge is bounded by a horn-shaped insulator. Current is fed to the water by metal electrodes at the small and large ends of the horn, as indicated in the figures. It is clear from these experiments that the equipotentials within the horn are indeed very closely spherical in shape. These experiments were done using a boundary shape identical with that of a trombone bell whose acoustical behavior we have studied extensively.

## III. Theoretical and Experimental Relations between Plane- and Spherical-Wave Formulations of Webster's Equation

We consider now the implications of our two representations for waves in a type of horn that is familiar both to musicians and to acousticians: This is the Bessel horn, whose geometrical radius $R$ varies hyperbolically with distance:

$$R(x) = B/(x)^{\alpha} \tag{9}$$

## III-A. The Plane-Wave Representation

If we follow tradition and assume plane waves, the horn function $U(x)$ for use in Eq. (4) takes on the form

$$U(x) = \alpha(\alpha + 1)/x^2 \tag{10}$$

Notice that our coordinate system is slightly unfamiliar. The coordinate origin is located at the large end of the horn, where the radius function $R(x)$ has a pole. A solution to Eq. (4) that fits the requirement that the pressure is small at the open end, $p(x = 0) = 0$ , as laid down in Sec. II-B, is

$$\psi(x) = \sqrt{x}\, J_{\alpha+1/2}(kx) \tag{11}$$

And, according to Eq. (3), the pressure distribution is then given by

$$P(x) = \left(\frac{\sqrt{x}}{R}\right) J_{a+1/2}(kx) \tag{12}$$

The discussion in Sec. II-B shows that at the blowing end of the horn we must have a maximum pressure amplitude for maintenance of oscillation, so that our second boundary condition is that $p(kL)$ = maximum. The first column of Table II presents the values $k_nL$ for favorable regeneration, which imply the corresponding values of $\omega_n$. [Table I, which gives measured values, is discussed later in this paper.] In this preliminary investigation we have ignored the complications introduced in an actual brass instrument by the presence of the mouthpiece cavity. This is a matter that W. T. Cardwell has discussed recently (*see* abstract in December 1966 *J. Acoust. Soc. Am.*).

## III-B. The Spherical-Wave Representation

Figure 1 shows the new coordinate system in its relation to the horn, where $z$ locates the position of a spherical wave front along the horn axis, measured from the same coordinate *origin* as that used to locate the position $x$ of the assumed plane-wave fronts. As before, $R(x)$ is the geometrical radius of the physical horn at the point $x$. A computer calculation was carried out for a family of horns to give, among other things, the value of the new horn function $U(z)$, for use in the wave equation:

$$\frac{\partial^2\psi_n(z)}{\partial z^2} + [k_n^2 - U(z)]\psi_n(z) = 0$$

(13)

Figure 4 shows the relation of $U(z)$ to the plane-wave version $U(x)$ based on Eq. (10), as calculated for the trombone bell whose shape was used in the electrolytic tank experiments. We notice that as $z$ gets large, $U(z)$ approaches $U(x)$, as it should. $U(x)$ has a singularity at the origin (which is connected with the discontinuity problems arising in the treatment of radiation), while $U(z)$ lacks this singularity. We notice also that the "turning point" defined by $k^2 = U(z)$ is farther away from the blowing end in the case of $U(z)$ than is the turning point $x_t$ [defined from $U(x)$. This implies that the predicted natural frequencies $\omega_n$ arising from calculations based on $U(z)$ will be lower than those based on $U(x)$. The reason for this result is to be found in the fact that Bessel-related functions of the general form given for $\psi$ in Eq. (11) tend toward a large-argument asymptotic form which is a sinusoid:

$$\psi_n(z) + \sin(k_n z - \phi_n) \quad (14)$$

A little computation shows that $\phi$ is well approximated by the quantity $k_n z_t$, and that as a result we can imagine replacing the actual horn by a cylindrical pipe of length $L_t$ given by either form of Eq. (15):

$$L_t = L - x_t \quad (15a)$$

$$L_t = L - z_t \quad (15b)$$

The boundary conditions for this hypothetical pipe are the following: At the blowing end (as before), pressure is maximum for favorable regeneration with a pressure-controlled valve mechanism; at the other end, pressure is a minimum, as in the traditional open-ended pipe.

Symbolically then the natural frequencies of a horn with a Bessel-type shape are given by a formula of the type

$$\omega_n \cong (2n - 1)\pi c/2L_t(\omega_n) \quad (16)$$

One of [Benade] us called attention to this fact last year [abstract, *J. Acoust. Soc. Am.*, 39(6):1220 (June 1966)], and pointed out that this implies that musically useful horns must be of such a shape that the length $L_{tn}$ belonging to the $n$'th mode must be related to that of the first mode in a manner that is approximated by the formula

$$L_{tn} = L_{tl} \cdot (sn - 1)/2n \quad (17)$$

We have apparently digressed from the mainstream of the argument, but in fact this aside on turning-point lengths has been injected to make easy the qualitative assessment of a part of the effect of small changes in the horn function on the natural frequencies. It is a matter of familiar experience that the plane-wave approximation to Webster's equation leads to overestimated normal-mode frequencies (as expected in the light of Rayleigh's principle), so that the lowering of the $\omega_n$'s by use of the spherical-wave representation indicates its suitability.

## III-C. Radiation from the Open End

If one takes the plane-wave approximation seriously, the radiation from a complete horn (which extends from $x = 0$ to some length $L$) is identically zero. This is because $\psi(x)$ vanishes at the origin as a result of the infinite value of $U(0)$. If the horn is not complete to the origin, so that $U(z)$ stays finite at the open end, we are tempted to gloss over the difficulties of connecting plane and spherical waves to make a brute-force estimate of the radiation.

Inside the horn, $\psi$ is of the order of magnitude of unity in its amplitude, and if the physical horn is of hyperbolic shape up to the point $x_0$ at its rim, we may use a small-argument expansion formula for $\psi$ to estimate its size outside the bell. In this way we get an estimate for the transmission T of sound energy through the "barrier" $U(x)$ extending from $X_t$ to $X_0$

$$T = \text{constant}\,[\psi(x_o)/\psi(\mathit{inside})]^2 = \text{constant}\,[kX_o^{\alpha+1/2}/1]^2\,k^{2x+1} \tag{18}$$

In other words, the gross trend is such that the radiation contribution to the half-power bandwidths of the various resonances should rise as the $(2x + 1)$ power of the resonance frequency. As we shall see, this is not consistent with the experimental measurement of radiation loss.

The radiation loss in the spherical-wave case is handled somewhat differently. We have a computed horn function $U(z)$, but no convenient analytical representation for it. In any event the exact calculation of the transmission through the barrier presented by $U(z)$ near the bell of a horn would be quite difficult. An escape from this problem may be found in the application of WKB methods, which are known to work well in dealing with similar problems of radioactive decay in nuclei. Referring to Fig. 4, let us define a pair of "turning points" $z_{ti}$ and $z_{t0}$ on each side of the barrier $U(z)$. The inside turning point $z_{ti}$ is the same one that we have made use of earlier, while the outside point $z_{t0}$ is the one on the side of $U$ that is toward the outer world. In both cases the turning points are calculated from the formula

$$U(z_t) = k^2 \tag{19}$$

The transmission coefficient $T$ for power through the barrier is then given by the formula

$$T = \exp\left\{-2\int_{z_{ti}}^{z_{t0}} \sqrt{U(z) - k^2}\, dz\right\} \tag{20}$$

**III-D Theoretical Basis of Experimental Work**

Because laboratory apparatus seldom is more than approximately similar to the idealized abstractions with which one deals in theory, we must provide ourselves with a few correction terms. In the experiments described below, we used a trombone bell attached to a long cylindrical pipe and compared the acoustical properties of the composite system with the properties of the very much more familiar cylindrical pipe when taken by itself.

When a section of horn is attached to a cylindrical pipe, there are two major effects that alter the normal-mode frequencies away from the values belonging to a complete horn whose total length is the same as that of the composite. In the spirit of the WKB (phase-integral) approximation to the wave equation, we find that the boundary conditions require us to specify that the total phase accumulated from one end of the horn to the other is equal to an odd multiple of $1/a$. This is familiar in the case of cylindrical horns, where the standing waves for our boundary conditions are such as to fit an odd number of quarter-wavelengths into the pipe length. If we alter a horn, so that in some region between $z_1$ and $z_2$ the function $U(z)$ is altered by an amount $\Delta U(z)$, then there will be an alteration $\Delta\phi$ in the total phase (*see* Morse and Feshbach, *Methods of Theoretical Physics*, for a thorough discussion of WKB phase-integral methods):

$$\Delta\phi = \int_{z_1}^{z_2} \left\{\sqrt{k_n^2 - (U + \Delta U)} - \sqrt{k_n^2 - U}\right\} dz \tag{21}$$

While the phase-integral approach is not very accurate in the rapidly varying part of the horn near the bell, we may use it safely to estimate the effect of replacing the slowly tapering small end of a horn by a cylindrical pipe. Let $\phi_{in}$ be the effect of changing the system in this manner, and write $\Delta U(z) = U(\text{cyl}) - U(\text{horn})$. The magnitude of $\phi_{in}$ is then given by

$$\phi_{in} = \int_{L}^{L-L_p} \left\{ k_n - \sqrt{k_n^2 - U(z)} \right\} dz$$

$$= \int_{L}^{L-L_p} \left\{ U(z)/2k_n \right\} dz \qquad (22)$$

Here $U(z)$ represents the horn function belonging to the missing portion of the complete horn, and $L$ is the length of the cylindrical pipe that replaces it. The sense of this phase shift is such as to lead to a lowering of the natural frequencies when the pipe is used.

There is a discontinuity at the junction of the horn with the pipe where the plane waves belonging to the pipe must join onto the slightly spherical waves in the horn. The equations for continuity show (if the pipe diameter is the same as that of the horn throat) that at the junction

$$[d(\log r_0)/dz]_{\text{pipe}} - [d(\log r_0)/dz]_{\text{horn}}$$

$$= [d(\log \psi)/dz]_{\text{pipe}} - [d(\log \psi/dz]_{\text{horn}} \qquad (23)$$

For convenience let us replace the coordinate $z$ by a new one measured from the input end of the system, that is, $y = L - z$. In this coordinate system the standing wave in the neighborhood of the junction is

$$\psi_{\text{pipe}} = \cos k_n y \qquad (24a)$$

$$\psi_{\text{horn}} = A \cos(k_n y + \phi_2 n) \qquad (24b)$$

$$\phi_{2n} = (\alpha/k_n y) \cos^2 k_n y \qquad (25)$$

Once again the sense of this phase shift is such as to lower the frequencies observed in the combined cylinder plus bell relative to the frequencies of a complete horn of the same length. This last correction formula is a cousin to an expression obtained by N. H. Frank for the strength of reflections produced at the junction of two tapered microwave lines (*see* Moreno, *Microwave Transmission Data*, pp. 53-54).

Finally, we have to make allowances for the fact that viscous and thermal interactions of a soundwave with the walls of the horn produce boundary-layer effects. For present purposes, the chief effect is the lowering of the velocity of sound in the pipe relative to that outside, so that the observed rosonance frequencies are lowered once again relative to those of an idealized horn. Kinsler and Frey (*Fundamentals of Acoustics*, p. 240) give the correction formula as

$$C' = C_0\left[1 - 1/a\sqrt{\mu/2\omega}\right] \tag{26}$$

Here $a$ is the radius of the pipe, and $\mu$ is the kinematic viscosity of air (approximately 0.156, cgs). In our experiments the effect should be integrated over the whole length of the air column, but since the correction is small and falls rapidly in the larger parts of the bell, we may simplify things considerably by assuming a correction equal to that belonging to a pipe of length $L_t$ whose radius is that of the cylindrical part of our system.

## IV. Experimental Results and Discussion

Experiments were carried out using a trombone bell very similar to the Boosey and Hawkes Imperial model (made by Rangarsons in New Delhi). The dimensions of this bell are given at the top of Table I, along with those of a 6-ft piece of copper tubing whose inside diameter (20 mm) is equal to the diameter at the small end of the bell. Resonance measurements were first made on the tube alone, driven at the closed end through a capillary of 0.5 mm ID and 25-mm length from a servo-controlled constant-sound-pressure source. A 1/2" diameter Brüel and Kjaer microphone and a specially made adapter sleeve formed a hermetic closure at the driving end of the tube. Along with the resonance frequency measurements, the 3-db bandwidths of the resonances were also measured. These pipe measurements served as a basis for checking the behavior of the composite system formed by joining the pipe to the trombone bell by means of an adapter. The observed resonance frequencies and half-power bandwidths of the various vibrational modes of the two systems are given in the body of Table I. The discussion in the following parts of this section is based on these data.

## IV-A. The Resonance Frequencies as Related to Theory

The first column of Table II presents the values of $k_nL$ calculated, using plane-wave theory, for a horn whose total length is equal to the sum of the lengths of the experimental bell and the length of the copper tube. The assumed flare parameter $\alpha$ and other properties of the horn are chosen to match those of the experimental bell. The second column of the Table II gives the corresponding values $k_n = \omega_n/c$ for the various vibrational modes. We shall compare these values of $k_n$ with those calculated from the experimental data and adjusted for the effects of the boundary layer and the replacement of a part of the ideal horn by a length of pipe. The column of Table II headed "$k_n$ observed" gives the quantity $(2\pi f_n/c')$ calculated from the observed resonance frequencies $f_n$, and the velocity of sound, $c'$, which is corrected for room temperature and the boundary layer effect [*see* Eq. (26) above, for the latter]. These "raw-data" values for the $k_n$'s are then adjusted upwards by application of the phase-integral correction $\phi_{n1}$ and the bell-to-pipe joint correction $\phi_{n2}$, as given in Eqs. (22) and (25). The corrected $k_n$'s, which are presumably suitable for comparison with the theoretical values, are obtained from the raw $k$'s by means of the following formula:

$$k_n^{\text{obs}}(\textit{corrected}) = \left[1 + \frac{|\Phi_{n1}| + |\Phi_{n2}|}{(2n - 1)(\pi/2)}\right] \qquad (27)$$

This form is based on the fact that the total phase change from one end of the standing wave to the other is $(2n - 1)\pi/2$ radians [*see* Eq. (21)]. The fourth column of Table II gives these corrected values, and the fifth column presents the fractional discrepancy $D_n$ between the plane-wave results and the experimental data as modified:

$$D_n = \frac{k_n^{\text{calc}} - k_n^{\text{obs}}(\text{corrected})}{k_n^{\text{calc}}} \qquad (28)$$

Let us now consider the relationship between theory and experiment as indicated by the mode-number dependence of $D_n$. Comparison between the prediction and the observation for mode 1 has not been made because of the inherent inadequacy of the phase-integral correction for the replacement of part of the horn by means of cylindrical pipe. Such a correction has been set up on the assumption that $U(z)k_n^2 \ll 1$, a clearly inadequate assumption for mode 1 when the corresponding value of $k^2$ ($= 5.3 \times 10^{-5}$) is compared with the range of values for $U(x)$ and $U(z)$ displayed in Fig. 4. As a matter of fact the turning point for this mode in a complete horn would fall well within the region that is replaced by the pipe. The accidental vanishing of $D_2$ is of little significance, and is probably due to experimental variability associated with measuring a low-Q resonance at low frequencies, compounded with the great sensitivity of the lower modes to the effects of temperature gradients on the local characteristic impedance within the bore [*see* abstract, Thermal perturbations in woodwind bores, *J. Acoust. Soc. Am.* 35(11):1901 (1963)]. The behavior here would be like that described for a clarinet).

The negative values for the discrepancies in modes 3 and 4 are probably attributable to the systematic tendency of the pipe-replacement correction to be large when applied in first order at low mode numbers. It is to be recalled that WKB methods in general are most suitable for vibrational states that have many nodes and loops. The sequence of discrepancies $D_4$ through $D_7$ is interesting and significant. We see clearly that the plane-wave approximation is beginning to overestimate the resonance frequencies. Reference to the graphs for the plane-wave and spherical-wave horn functions $U(x)$ and $U(z)$ plotted for our bell in Fig. 4 shows that the classical turning point, and hence the "effective length," of a horn is progressively farther from the blowing end as $k_n^2$ rises in the case of $U(z)$ in comparison with the plane-wave function $U(z)$. This suggests that we can remove or reduce the discrepancies $D_n$, by replacing $U(x)$ by $U(z)$ in the theoretical calculation. This is not easily done, because it requires a numerical integration of the wave equation based on the tabulated values of $U(z)$. However, we may make a quick estimate by simply comparing the magnitudes of the various $D$'s with quantities $\delta_n$ defined as follows:

$$\delta_n = \frac{L_{tn}(\text{spherical}) - L_{tn}(\text{plane})}{L_{tn}(\text{spherical})} \tag{29}$$

These are tabulated in column 6 of Table II. The $a$'s give the estimated fractional lowering of the resonance frequencies due to the change in "effective length" produced by the change from a plane-wave representation to one in which spherical waves are assumed. It is at once apparent that the *trend* of the modification is in the right direction. The higher modes are lowered progressively by going to spherical waves. The accuracy of the experiment, and the uncertainties of the bore correction discussed earlier, prevent useful discussion of the fact that there is still a residual discrepancy. We hope to pursue this matter further, with better temperature control and a more accurate system of computational corrections.

## IV-B. The Behavior of the Radiation Damping

Let us turn now to the question of radiation damping in the horn. As has been mentioned earlier, Table I gives the observed half-power bandwidths $g_n$ of the various modes of both the pipe, and the pipe plus trombone bell. As is well known, the boundary-layer interactions of a sound wave in a cylindrical pipe give rise to a dissipation, and hence $g_n$ rises as the square root of frequency, in the absence of radiation, so that we must subtract out this contribution from the total $g$ observed in the laboratory:

$$g_{\text{total}} = g_{\text{boundary layer}} + g_{\text{radiation}} \tag{30}$$

The theoretical expressions [*see* Kinsler and Frey, and Eq. (26), for example] for boundary losses are well verified experimentally by the behavior of our pipe when taken alone in the frequency range of interest. Two methods were used in estimating the damping: the direct method, where the 3-db points on each side of the resonance are measured, and the method based on the amplitude ratios between the resonances and antiresonances. This latter method is sensitive to the finite impedance of the capillary tube that drives the air column. The two methods agree with one another and with theory when corrections are made for the capillary. The small-circle data points in Fig. 5 show the observed resonance widths for the combined pipe and horn, plotted in reduced form as $g/(f)^{2/3}$. At low frequencies, where little radiation is expected, we find that the trend of these points is horizontal, in agreement with expectations based on the theory of boundary-layer losses. These points are also in agreement with a calculation based on the observed losses in a pipe alone, as adapted to the composite system. Above about 500 Hz we find that the damping width for the composite horn rises very sharply, as radiation begins to become important. The triangular data points in Fig. 5 show the values for $g_{radiation}$ calculated by subtracting the extrapolated boundary-layer losses from the observed total widths. It is convenient for future analysis to notice that the slope of the line drawn through the triangular points on the log-log graph has a slope 3.9, which indicates that $g_{radiation}$ *measured* for our bell varies as the $(3.9 + 0.5 = 4.4)$ power of the frequency, as noted in the top right section of the graph.

In Sec. III we made an estimate of the transmission coefficients $T$ through the barrier formed by $U(x)$ and $U(z)$. These transmission coefficients are proportional to the radiation widths by way of the standing-wave energy storage properties of the horn. The exact proportionality constant is a little tedious to estimate analytically for our combined horn in the plane-wave case, and is very difficult to do except numerically for the spherical-wave case. At present we need not evaluate these proportionality constants, since they are known not to vary with frequency to any great extent.

The dashed line drawn across the middle of Fig. 5 has the slope expected (via the transmission coefficient) for the frequency variation of $g_{radiation}$ upon the assumption of plane waves. We have already seen [in Eq. (18)] that the transmission coefficient varies as the $(2\alpha + 1)$ power of the frequency, where $\alpha$ is the horn parameter. In the case at hand the plotted slope is then $2.26 - 0.5 = 1.76$. It is clear that the plane-wave representation predicts a very much slower rise in transmission than is experimentally observed.

The square points plotted in Fig. 5 show the results of a phase-integral calculation of the transmission coefficient based on spherical waves. The formula from which these points are obtained is given in Eq. (20). Once again it appears that a straight line may usefully be drawn through the computed points, whence we deduce that the variation of $g_{radiation}$ should go as the 3.5 power of the frequency. On the graph, the plotted slope is therefore 3, in remarkable agreement with the experimentally observed slope.

## IV-C. Some Serious Anomalies

So far the relation of theory and experiment has seemed to be good, particularly regarding the improved accuracy of the spherical-wave approximation. There are some indications however that all is not so simple. First we notice that it is possible to get resonance behavior at frequencies above that at which the calculated transmission coefficient becomes equal to unity! In the spherical-wave approximation we find that the horn ceases to store wave energy at frequencies above $f_{max}$ given by

$$(2\pi f_{max}/c)^2 = U_{max}(z) \tag{31}$$

For the bell in question $f_{max}$ falls at about 655 Hz, while the frequencies of the 10th through the 14th modes are clearly measurable (though these are above 700 Hz), and the resonance peaks are easily seen on a chart recording of the resonance curve. It is obvious that something is occurring that serves to raise the actual height of the barrier above the calculated value of $U(z)$. Furthermore, this additional height must be found only in the immediate neighborhood of $U_{max}(z)$, otherwise the good agreement of the spherical-wave normal modes with experiment will be destroyed. As a matter of fact, as we have seen, comparison of the discrepancies $D_n$ between the observed and calculated $k_n$'s with the expected shift $\delta_n$ produced by going from $U(x)$ to $U(z)$ implies that one might even hope for something that reduces the "true" value of $U$ relative to $U(z)$ in the region below $U(z) = 0.01$, which corresponds to the value of $k^2$ for mode 8!

If it were not for this fact, we might simply revert to some modification of the plane-wave approach. Considerations based on the ideas of the preceding paragraph were stimulated the following line of thought. Because the *curvature* $\partial^2\psi/\partial x^2$ of the reduced wave-function $\psi$ is equal to $(k^2 - U)$, it seems reasonable to seek to plot out the locus of $U$ by using a probe microphone run in along the axis of the horn to measure out the axial pressure distribution in a standing wave at various frequencies. One would only have to convert the observed $p(x)$ into the equivalent $\psi$, and then look for the axial location of the inflection point of $\psi$, which is also the turning point. The ability to discover the actual shape of $\psi$ itself, and thus to deduce the nature of $U$, is something of which the practitioners of quantum mechanics can become quite envious. The experiment was carried out, and at the same time it was verified by direct measurement that the wave fronts of the standing wave within the bell were very accurately spherical for the half-dozen lowest modes of the horn.

Figure 6 presents a decibel plot of the observed pressure amplitudes along the axis of the bell when driven at various frequencies. The individual crosses represent individual readings, so that there is little room for ambiguity. As is usual in a decibel plot, only the *shapes* of the various curves have any significance: they are displaced one over another to keep the individual curves distinct. For frequencies below about 600 Hz, the curves are smooth, and very much as one would expect. Above this, however, there are distinct downward cusps in a region well within the peak of the horn function $U(z)$. These cusps and the greatly increased curvature of the contours on either side give strong evidence of an interference phenomenon taking place between the ordinary wave and some coupled wave within a new mode of propagation. At the present time we interpret this as suggesting that a higher-order ($\nu \neq 0$) mode coming into being in the expanding part of the bell, which is coupled to, and interferes with, the original ($\nu = 0$) propagation mode in the region of appreciable $U(z)$.

Presumably the trapping of energy at high frequencies comes about by an additional component of $U(z)$ that acts on this higher-order wave to keep it in the system, while the ordinary component ($\nu = 0$) escapes freely. This viewpoint has progressed somewhat beyond the realm of idle speculation, and evidence of the possibility of such behavior is visible in our initial crude attempts to obtain the horn equation by a generalization of the spherical-wave equation. The additional component of $U(z)$ comes in the way the centripetal potential does in the radial Schroedinger equation, so that it will look something like

$$U_p(q) = \text{constant}/q^2(z) \tag{32}$$

Here $q$ is approximately equal to the radius of the spherical wave front located at $z$. $U(q)$ is clearly a function that peaks in the general neighborhood of the maximum of the original $U(z)$, and falls rapidly on each side to zero at large distances.

The next matter to be investigated experimentally as a means for testing this speculation will be to seek additional evidence of interference patterns in the bell, with special emphasis on the search for signs of more or less horn-shaped nodal surfaces within the bell, part way out from the axis to the walls of the horn.

## IV-D. Conclusion

This report was written partly as an attempt to organize our own thinking, and partly as a means for informing our coworkers at other laboratories of the ideas we have had and of the experiments we have done. It is not intended that this draft document be considered by others as being more than an interim account of work in progress. It is our plan to submit a more formal account of much of this work for publication in *J. Acoust. Soc. Am.*, and other parts will be pursued as time and energy permit. The comments, criticisms, and suggestions of interested people are earnestly solicited.

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Note: These pages, reporting on work done when E. Jansson was a postdoctoral visitor in Benade's lab at Case Institute, with a few later changes by Benade, were typed onto a ditto master by Benade for circulation to the people he knew might find the work of interest. Virginia Benade has retyped the faded purple copy, doing light editing to correct a fair number of typos. Peter Hoekje (who earned his PhD in the same lab) is checking the accuracy of the typing, especially of the mathematical expressions. Contact Virginia Benade (216/752-9453 or vwb@po.cwru.edu) to see how soon the corrected version will be available.

[6 pp. of data and figs. hand drawn by Eric Jansson under Benade's supervision]

Table I. Measured values of resonanse frequencies $f_0$ and bandwidths g. Calculated bandwidths depending on radiation loss. Theoretically calculated transmission T.

Pipe: inner diam 2.00 cm length 184 cm.

Bell: Radius $R = \frac{B}{x^{\alpha}}$ $B = 12.80\ cm^{\alpha}$

$\alpha = 0.63$

$x_0 = 1.3$ cm

Length = 54.4 cm

| pipe only | | |
|---|---|---|
| n | $f_0$ Hz | g Hz |
| 1 | 45.46 | 2.93 |
| 2 | 138.8 | 4.23 |
| 3 | 231.7 | 5.28 |
| 4 | 325.5 | 6.46 |
| 5 | 419.0 | 7.56 |
| 6 | 511.9 | 8.05 |
| 7 | 605.6 | 8.46 |
| 8 | 689.9 | 9.79 |
| 9 | 793.1 | 10.6 |
| 10 | 887.1 | 11.3 |
| 11 | 979.8 | 12.0 |
| 12 | 1072 | 12.8 |
| 13 | 1167 | 13.7 |
| 14 | 1262 | 13.1 |

| pipe + bell | | | bell only | theoretical calc. |
|---|---|---|---|---|
| n | $f_0$ Hz | g Hz | g Hz | T |
| 1 | 40.3 | 2.14 | ~0 | * |
| 2 | 121.0 | 4.48 | ~0 | 0.00295 |
| 3 | 199.7 | 4.79 | ~0 | 0.0143 |
| 4 | 274.3 | 5.44 | ~0 | 0.0308 |
| 5 | 343.0 | 6.62 | ~0 | 0.0730 |
| 6 | 412.0 | 7.44 | ~0 | 0.152 |
| 7 | 485.7 | 7.56 | ~0 | 0.311 |
| 8 | 556.0 | 9.00 | .70 | 0.487 |
| 9 | 625.3 | 12.0 | 3.18 | 0.848 |
| 10 | 694.6 | 15.4 | 6.11 | over the barrier |
| 11 | 765.6 | 17.0 | 7.22 | |
| 12 | 838.8 | 23.6 | 13.3 | |
| 13 | 906.1 | 32.5 | 21.8 | |
| 14 | 975.9 | 39.7 | 28.6 | |

Temp +25.3 — +27.0 °C +27.0 — +27.7 °C

* outside the limits of used formulas

## Table II. Calculated and observed values of $k_n$.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $n$ | $k_n L$ calculated | $k_n$ calculated | $k_n$ observed | $k_n$ obs. corrected | $D_n$ | $\delta_n$ |
| 1 | 2.00 | .00834 | .00728 | * | * | * |
| 2 | 5.51 | .0230 | .0219 | .0230 | .000 | — |
| 3 | 8.72 | .0364 | .0361 | .0377 | −.036 | — |
| 4 | 11.9 | .0495 | .0496 | .0507 | −.024 | — |
| 5 | 15.1 | .0627 | .0620 | .0622 | +.008 | +.002 |
| 6 | 18.2 | .0759 | .0745 | .0746 | +.017 | +.003 |
| 7 | 21.4 | .0890 | .0874 | .0874 | +.022 | +.005 |
| 8 | 24.5 | .102 | .101 | .101 | +.020 | +.007 |
| 9 | 27.6 | .115 | .113 | .113 | +.017 | +.011 |
| 10 | over the barrier in the case of spherical waves | | | | | |

* outside the limits of used calculation formulas.

$$D_n = \frac{k_n^{calc} - k_{n_{corr}}^{obs}}{k_n^{calc}}$$

$$\delta_n = \frac{L_t^{(z)} - L_t^{(x)}}{L_t^{(z)}}$$

EJN

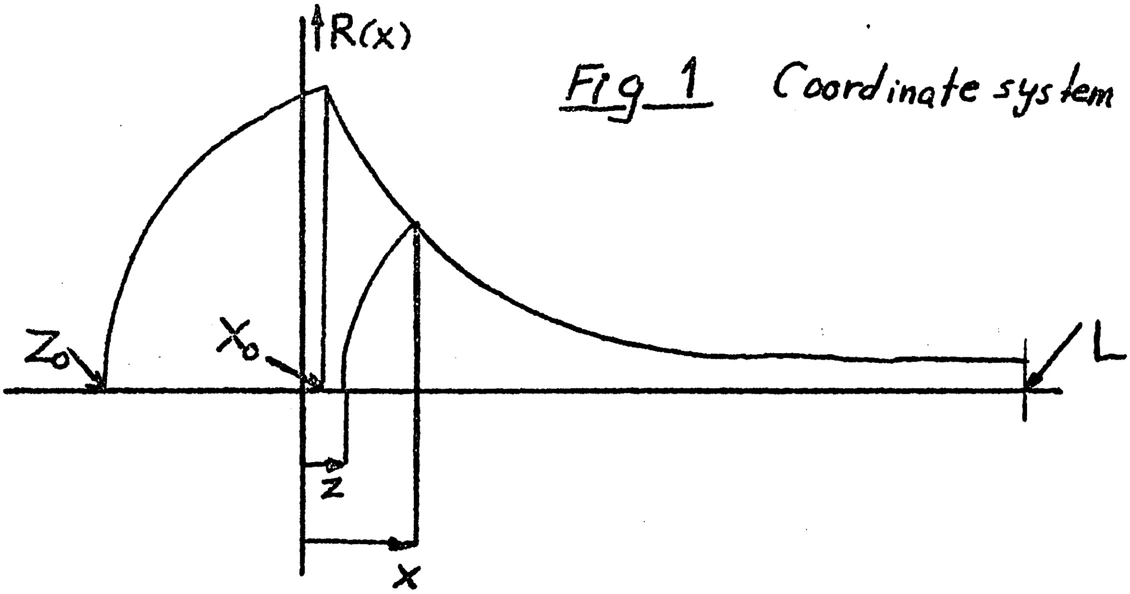


Fig 1 Coordinate system

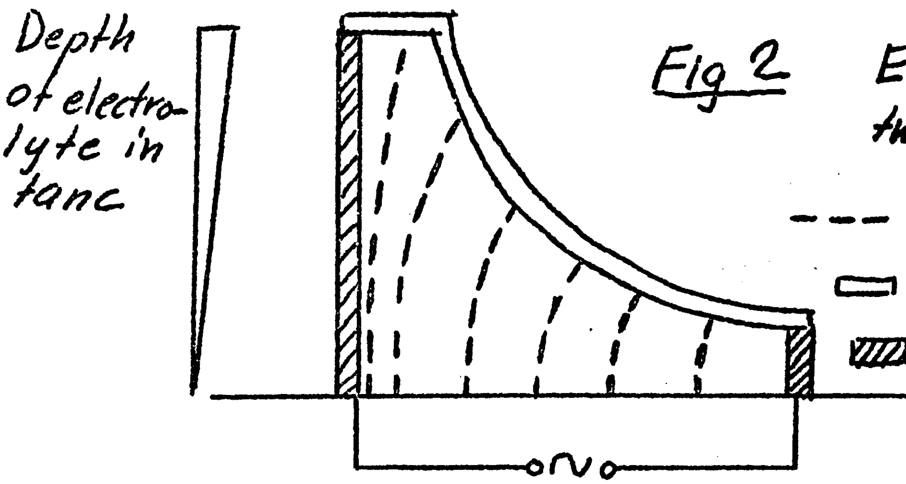


Fig 2 Electrolyte tanc two plane electrodes

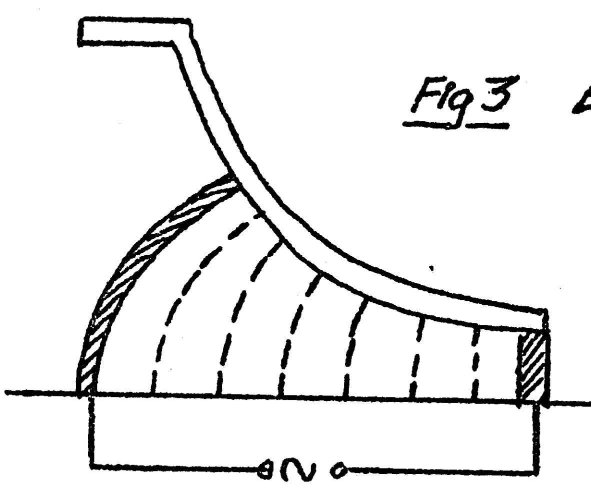

Fig 3 Electrolyte tanc one plane and one circular electrode

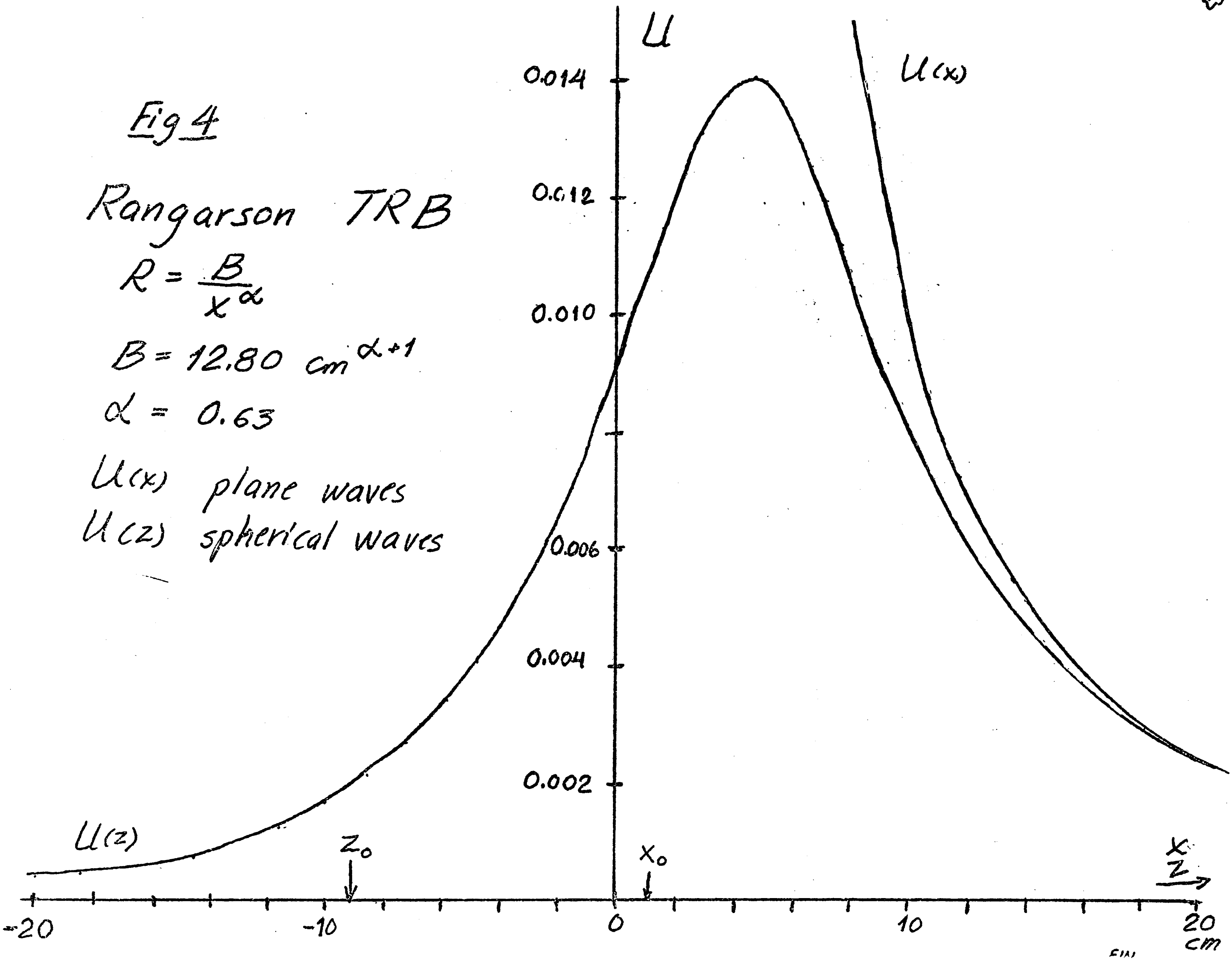
Rangarson TRB
R = B / x^α
B = 12.80 cm^(α+1)
α = 0.63
U(x) plane waves
U(z) spherical waves
U
U(x)
U(z)
0.014
0.012
0.010
0.006
0.004
0.002
z₀
x₀
x
z
-20
-10
0
10
20
cm

Fig 4

# Transmission - Radiation losses

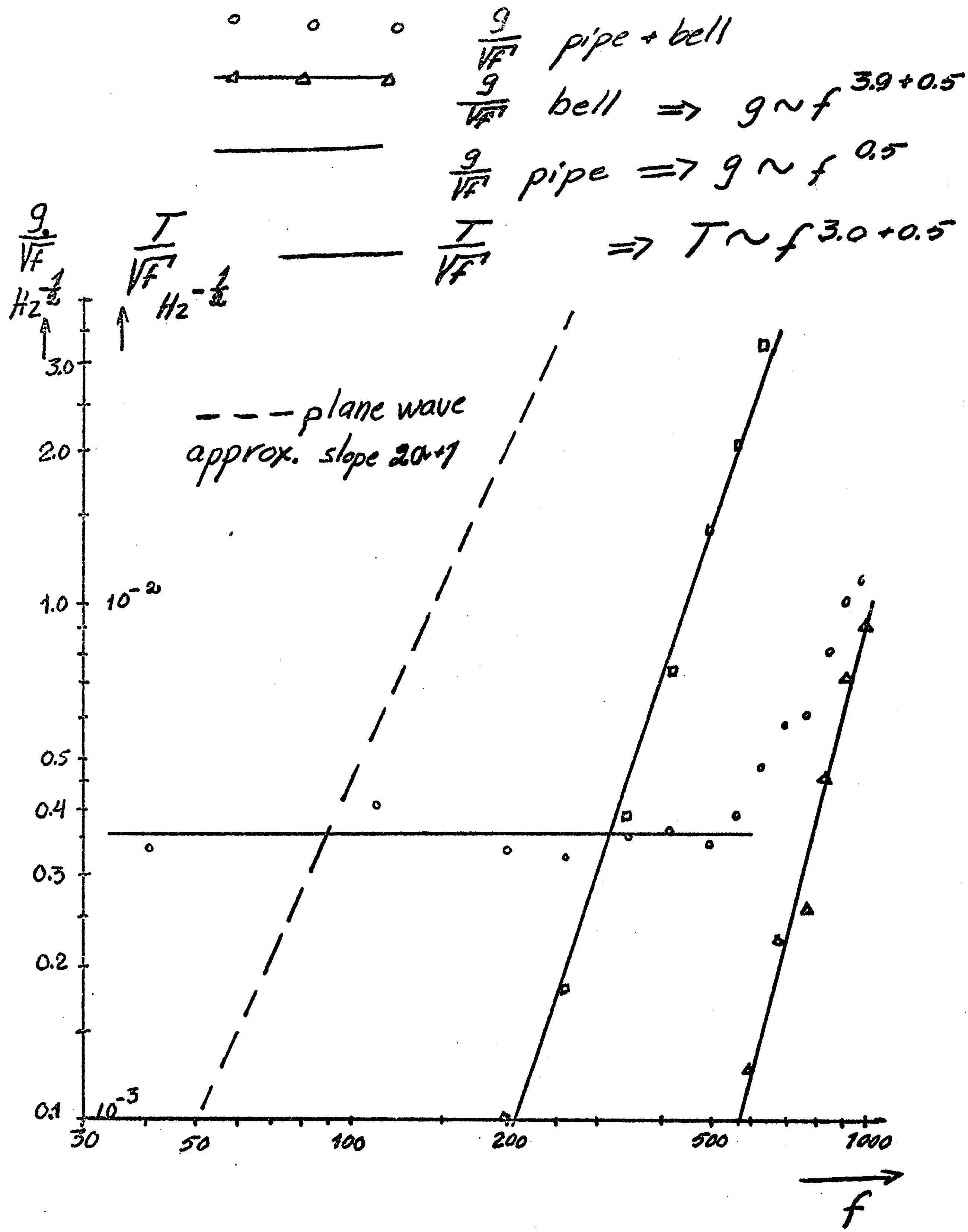


Fig 5.

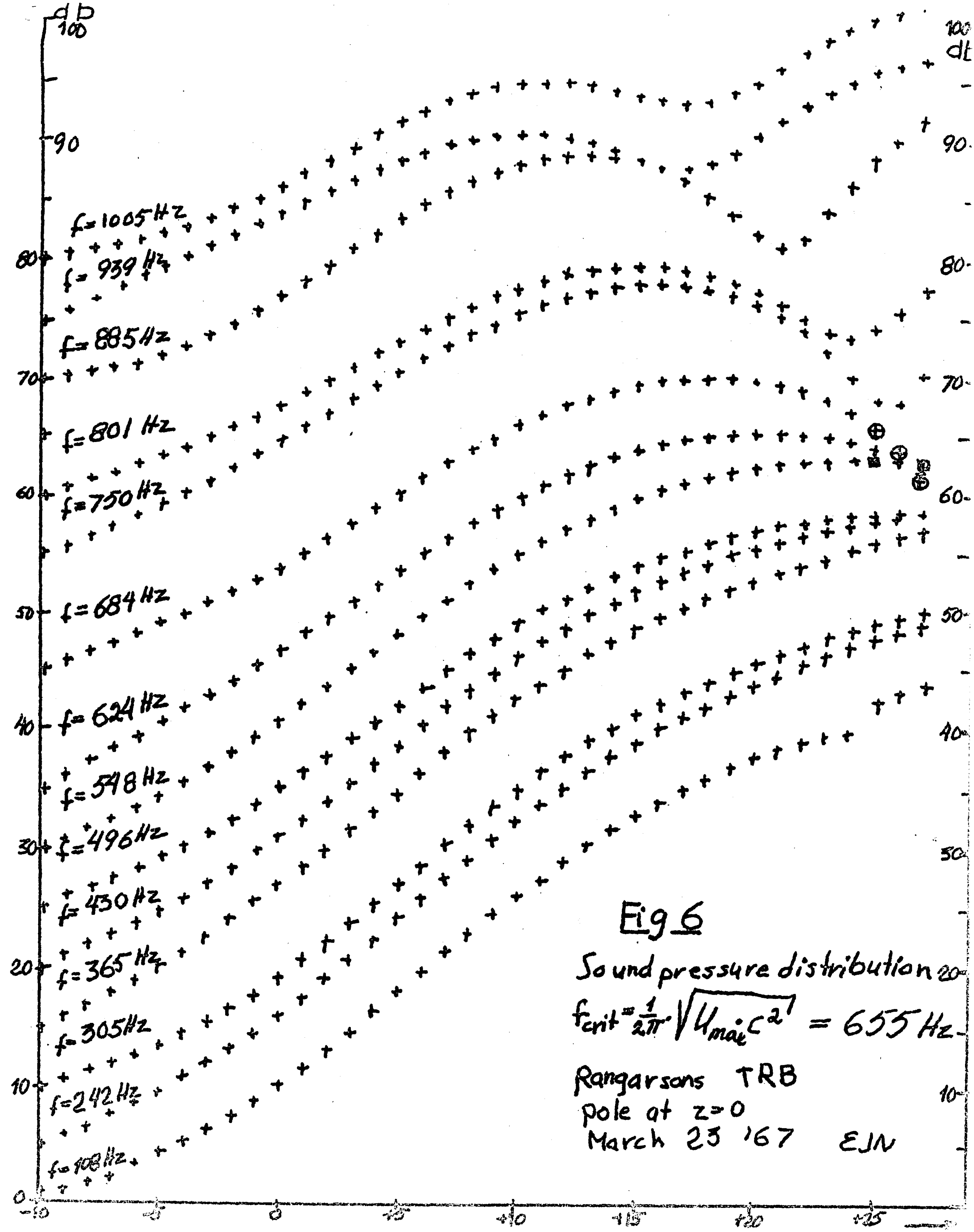


Fig 6

Sound pressure distribution

$f_{crit} = \frac{1}{2\pi}\sqrt{U_{mai} c^2} = 655\ Hz$

Rangarsons TRB
pole at z=0
March 23 '67 EJN