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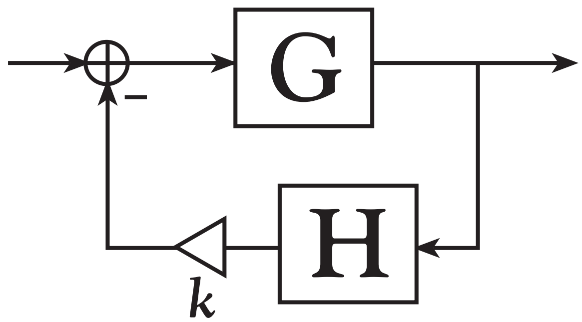
Applying Root-Locus Techniques to the Analysis of Coupled Modes in Piano Strings

- The Root-Locus Analysis Method
- Loaded N-Port Junction String Coupling
- 2-Mode Coupling
  - Root-Locus
  - Compare with Weinreich, etc.
- 2-String Coupling
- 3-Mode Coupling
- 3-String Coupling
- Conclusion

Originally presented at the Conference of the Acoustical Society of America, Hawaii, 1996. Also to be presented at the International Computer Music Conference, Greece, 1997

Root-Locus Analysis



Transfer Function:

$$\frac{G}{1+kGH}$$

Poles:

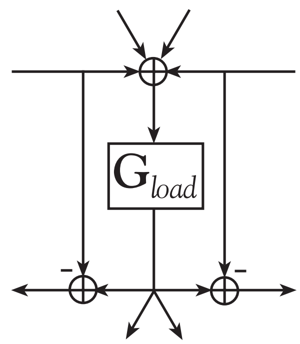
$$1+kGH=0 \Rightarrow GH=-\frac{1}{k} \quad k \text{ real}, k>0$$

Let $\frac{N}{D}=GH$**, so that:**

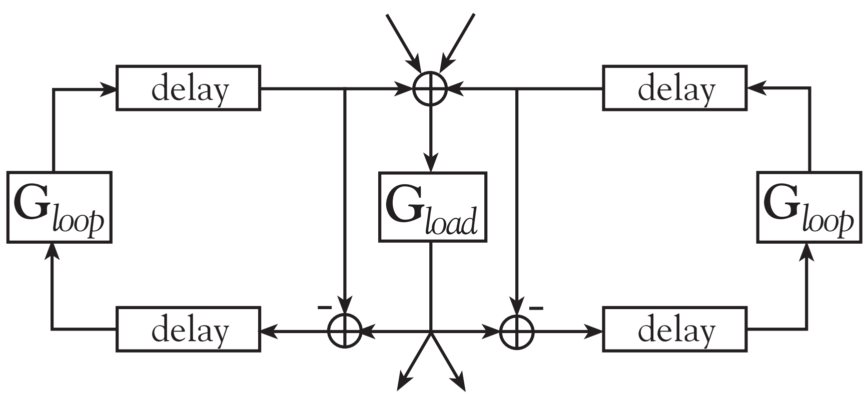
$$D+kN=0$$

- **At** $k=0$**, the system poles are at the poles of** GH
- **As** $k \to \infty$**, the system poles go to the zeros of** GH
- **The complete root locus:** $\{s : \angle G(s)H(s)=\pi\}$

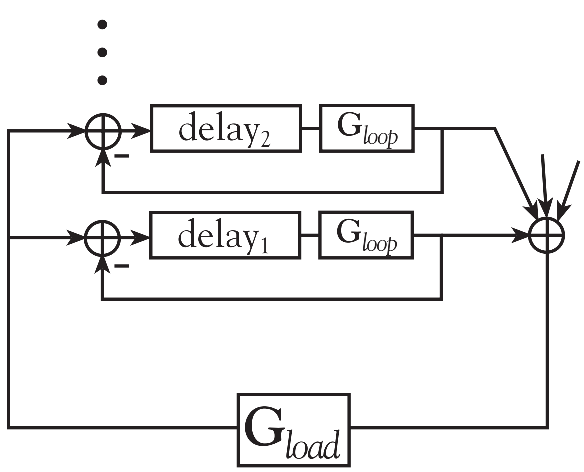
N-Port Junction String Coupling



An N-Port Junction

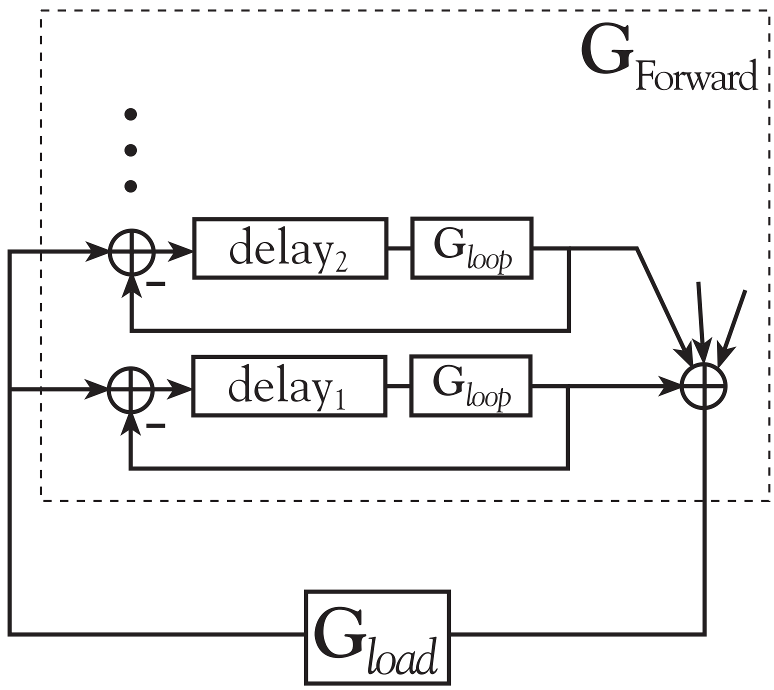


Coupled Strings



Rearranged to Root-Locus Form

N-Port Junction String Coupling



For simplicity, let $G_{\mathbf{loop}}(s) = -a$, then:

$$G_{\mathbf{Forward}(s)} = \sum_n \frac{-a\,e^{-sT_n}}{1 - a\,e^{-sT_n}}$$

Poles of each term:

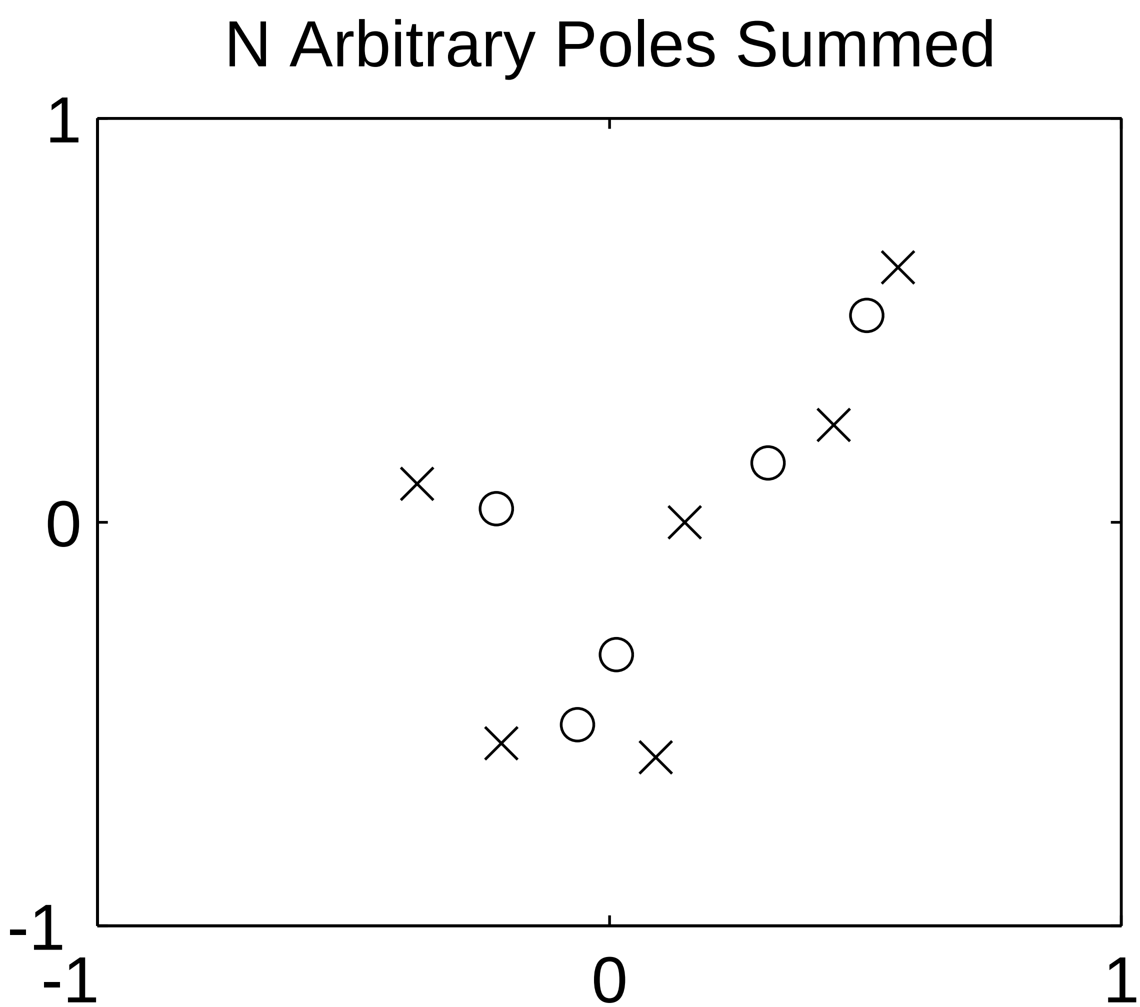
$$s = \frac{1}{T_n}(\ln a + ik\pi) \qquad \textbf{for } k = 0, \pm 2, \pm 4, \ldots$$

Note that we do not define any input or output. Pole locations are not affected by input/output choices. The 'zeros' we will talk about are the zeros of the loop transfer function.

Zeros of Summed Poles

- Where do the zeros land?
- Quick answer: "In between the poles"
  - Not completely true

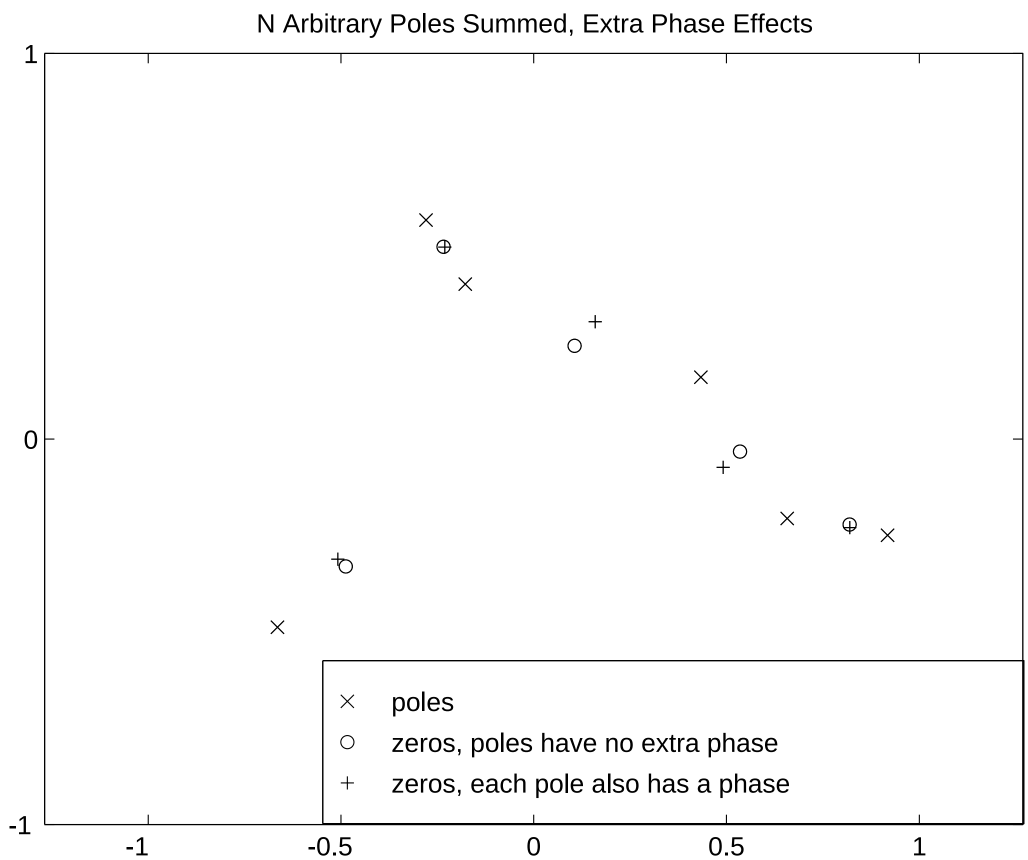
$$G(s) = \sum_n \frac{1}{s - p_n}$$



Zeros of Summed Poles, Phase Effects

- **Extra phase in the poles (which happens in the string case) moves the zero locations around.**

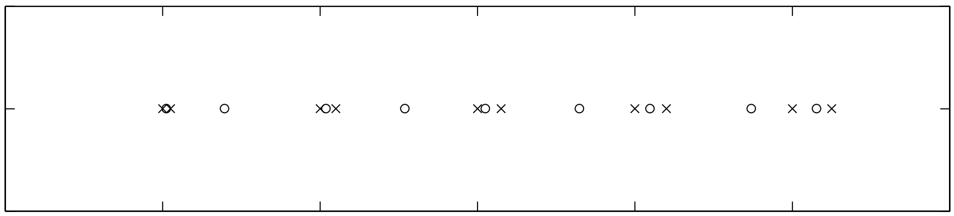
$$G(s) = \sum_n \frac{e^{i\phi_n}}{s - p_n}$$



Zeros of Summed Poles, Strings

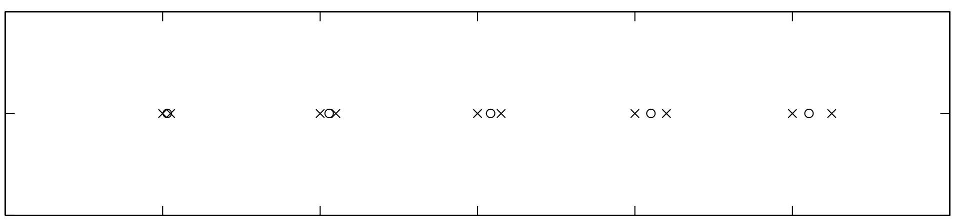
Summing two sets of poles: the *zeros* land between the poles.

$$\sum_i \frac{1}{s-p_i} + \sum_i \frac{1}{s-q_i}$$



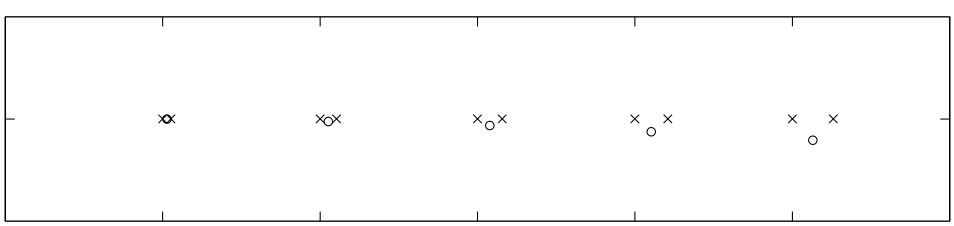
A single string is more like a product of its poles, so the sum of two strings has half the number of *zeros*:

$$\frac{1}{1-e^{-sT_1}} + \frac{1}{1-e^{-sT_2}} \approx \prod_i \frac{1}{s-p_i} + \prod_i \frac{1}{s-q_i}$$



The phase of the strings moves the *zeros*:

$$\frac{-e^{-sT_1}}{1-e^{-sT_1}} + \frac{-e^{-sT_2}}{1-e^{-sT_2}}$$



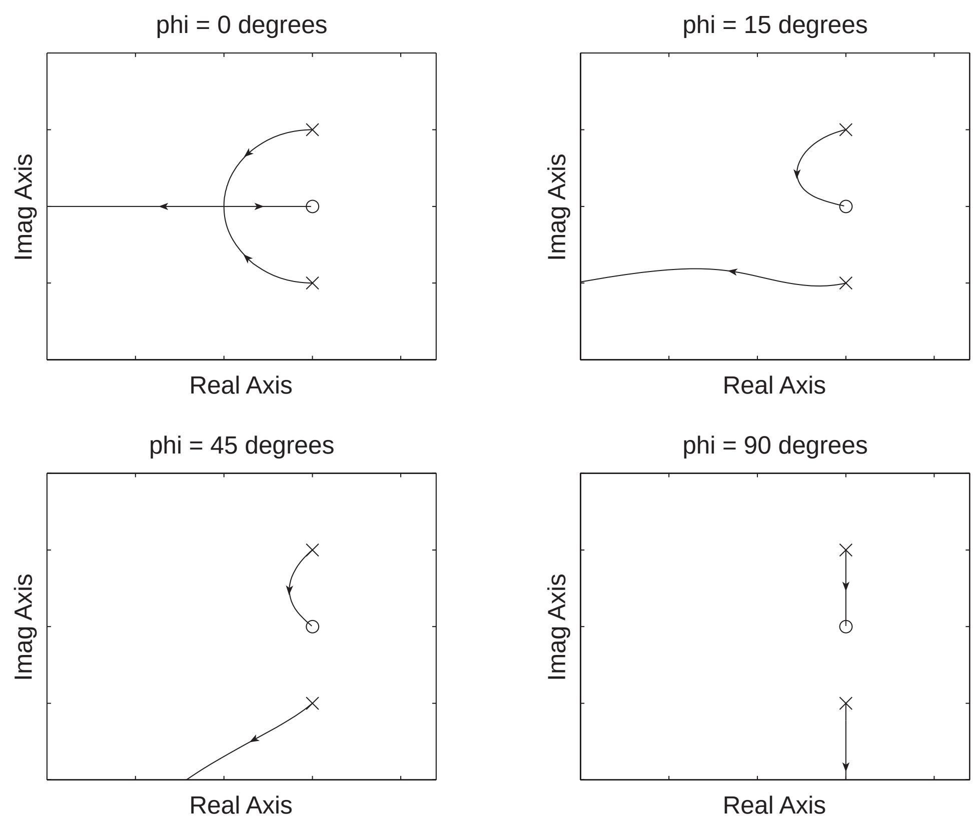
Basic 2-Summed-Pole Root-Loci

Root Loci in loop gain k

In this case, k can be interpreted as the mode coupling magnitude (the 'amount of coupling').

$$G(s) = e^{i\phi}\left(\frac{1}{s-p_1}+\frac{1}{s-p_2}\right)$$

We will call ϕ the 'coupling angle', which can be related to the coupling impedance. ϕ has the effect of 'rotating' the root-locus.

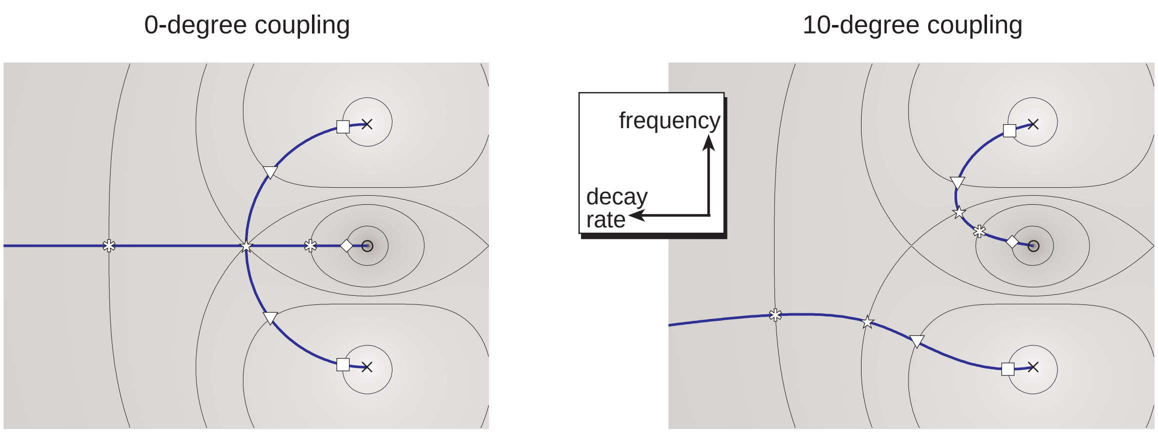


As coupling increases, the coupled-pole locations move along the root loci in the directions of the arrows. Note that the *shape* of the root locus is independant of the distance between the poles, only the effective coupling magnitude is affected.

Coupling Magnitude

We plot the Root-Locus on top of the magnitude of the loop transfer function ($|GH|$). The magnitude condition of the root-locus ($|GH| = \frac{1}{k}$) implies that the closed-loop poles will end up at the intersection of the $\angle(GH) = \pi$ and $|GH| = 1/k$ contours. Thus, the coupling magnitude k controls the character of the coupling: at low coupling, the poles stay near their open-loop locations and beat against each other; at high coupling, the poles go to similar frequencies, but with strongly different decay rates (the 'two-stage' decay).

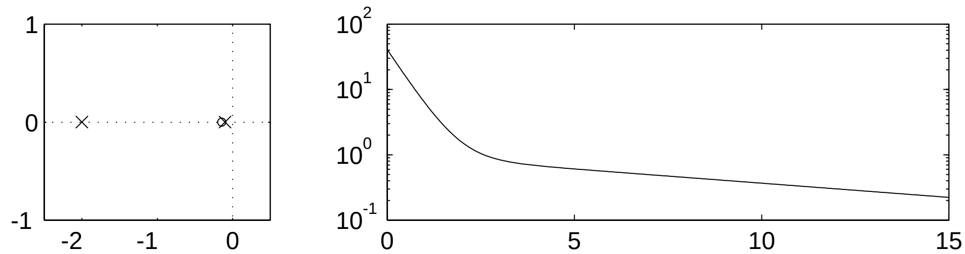
Root-Locus, with coupled-pole locations for a few values of coupling magnitude



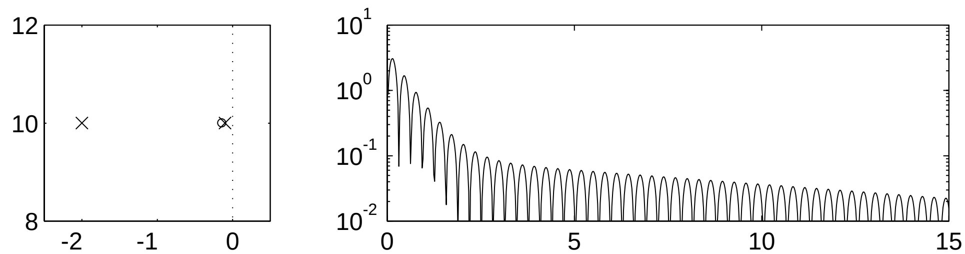
The distance between the open-loop poles has a large effect on the coupling: if the poles are close, the system will enter two-stage decay at lower values of coupling magnitude. This will have a large effect in detuned strings, where at low harmonics, the two strings' harmonic poles are close, and at high harmonics, they are further apart. Thus, a given value of coupling can cause two-stage decay at the low harmonics, but just cause beating at the high harmonics.

Decay and Pole Location

Sum of two real poles:

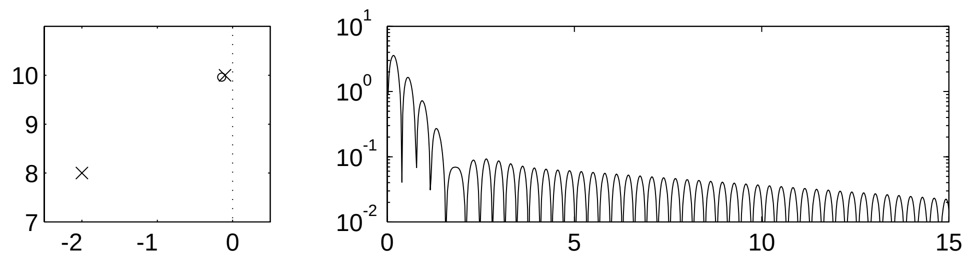


Sum of complex poles:

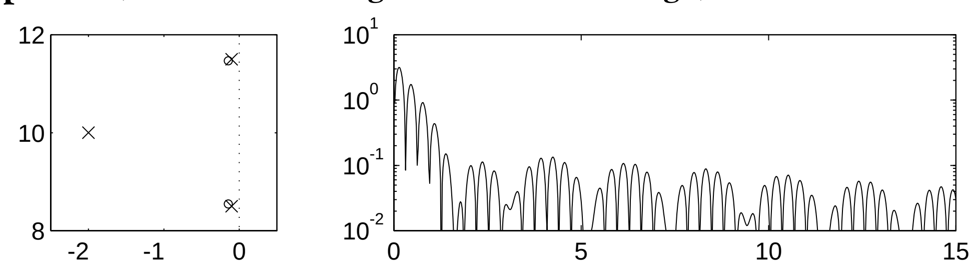


Sum of rotated complex poles (like complex coupling):

Note that the frequencies may beat together near crossover

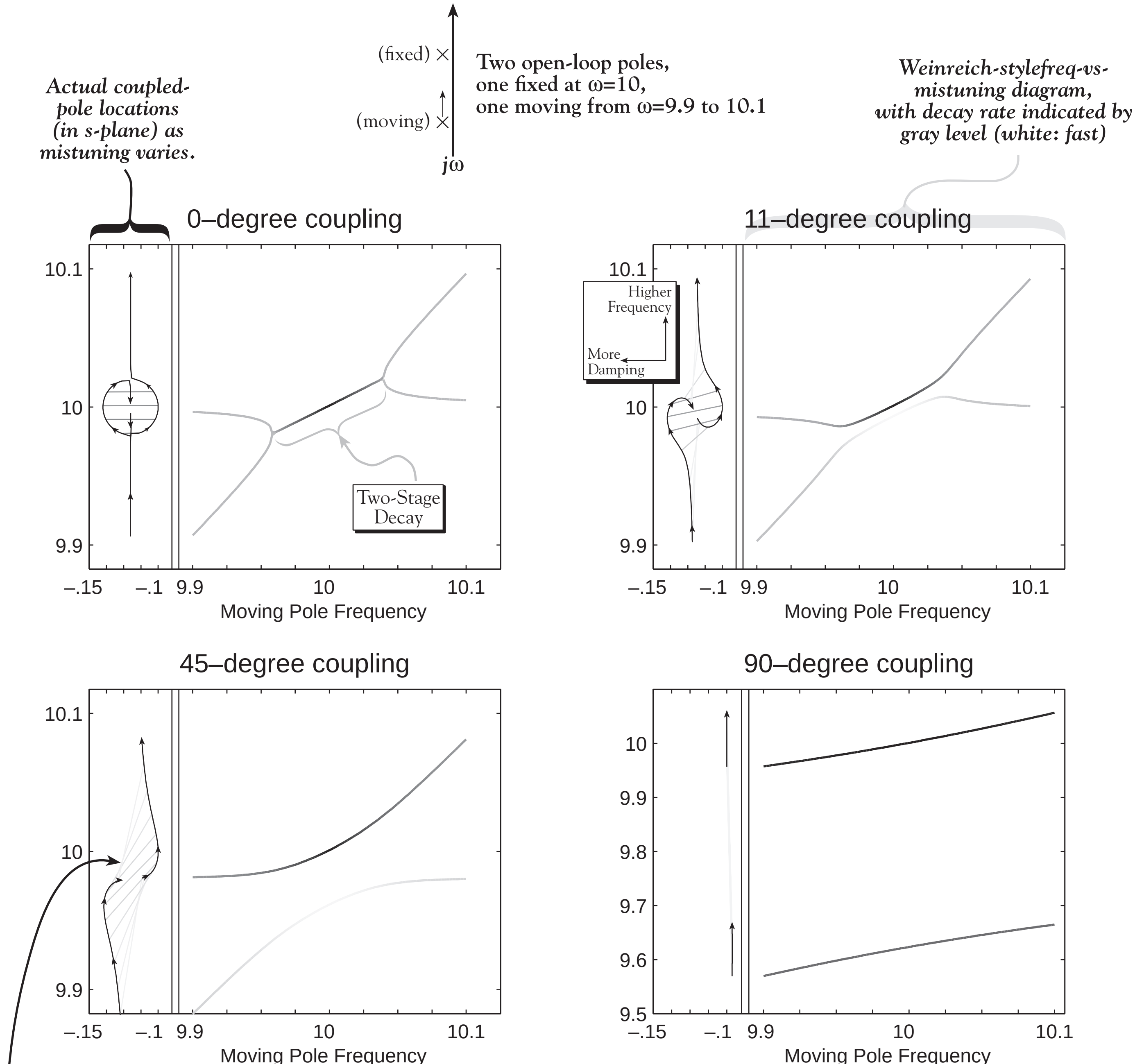


Three poles (note the beating in the second stage):



Comparison with Weinreich's Frequency vs. Mistuning Diagrams

(fixed coupling coefficient)

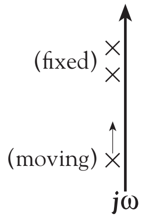


These lines connect poles of same open-loop mistuning. The closer to horizontal, the closer to black (the closer to standard "two-stage" decay).

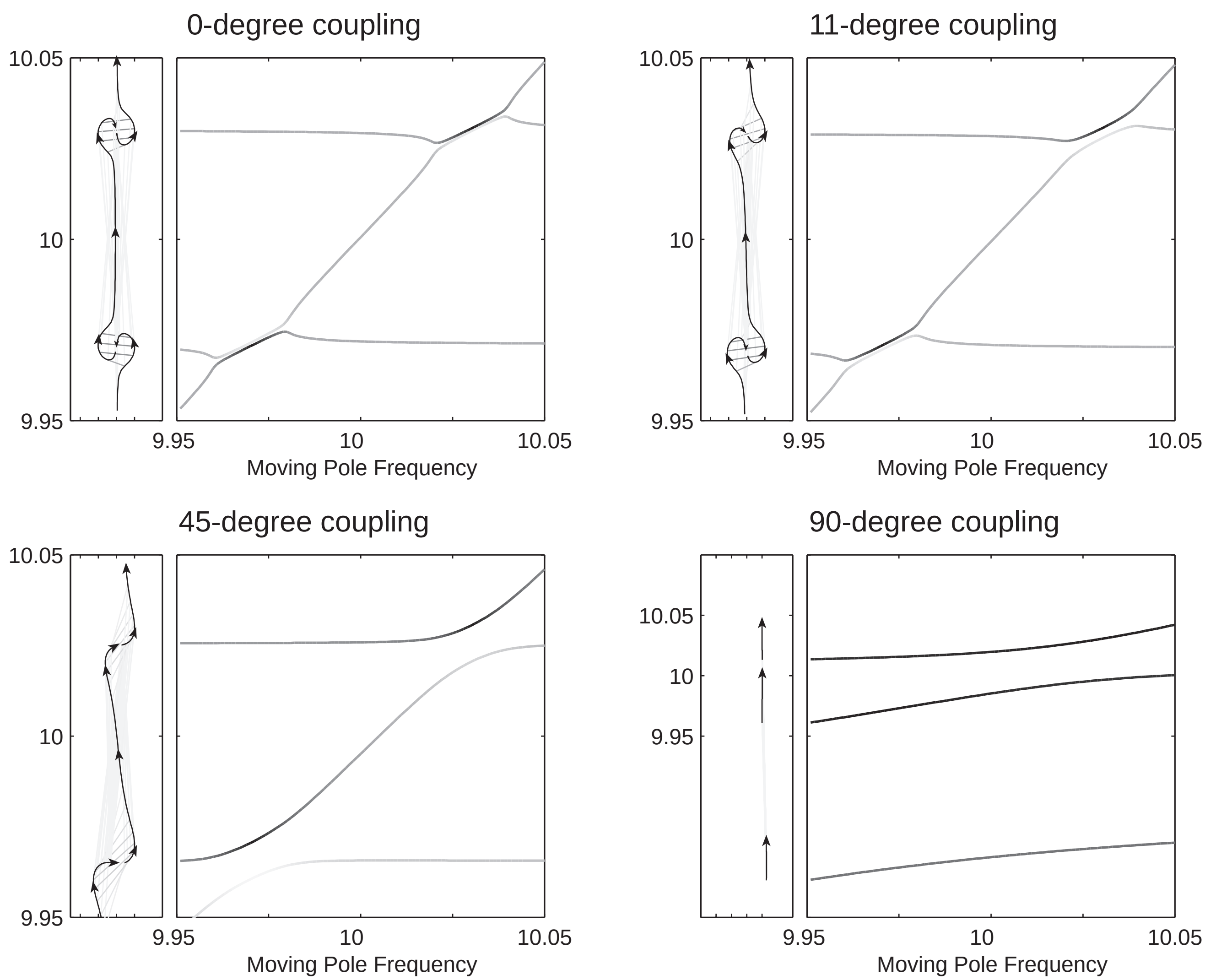
These diagrams (the ones on the left of each plot) correspond roughly to root-loci with mistuning as the free variable rather than coupling magnitude. In these root-loci, the coefficients of the closed-loop transfer function are not affine in the variable (as they are in k), so we will tend to see different patterns than in the 'standard' root loci.

It is interesting to note that the range of decay rates is independant of the coupling angle. The angle merely changes the relative freqencies of the poles at 'full coupling', keeping their rates the same. In terms of two-stage decay, this keeps the envelope of the two stage decay the same, but effects the beating pattern under the envelope.

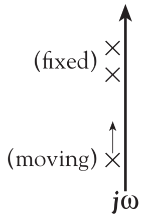
Next: we will look at 3-pole coupling. Unlike the two-pole case, where changing the distance between the poles has the same effect as changing the coupling magnitude, the presence of a third pole complicates the situation. Now, there will be more cases to look at. We will look at the following cases: (1) we will compare weak and strong coupling, and (2) we will compare the mistuning of just one pole and mistuning two poles at once. This gives us four combinations on the next four pages.



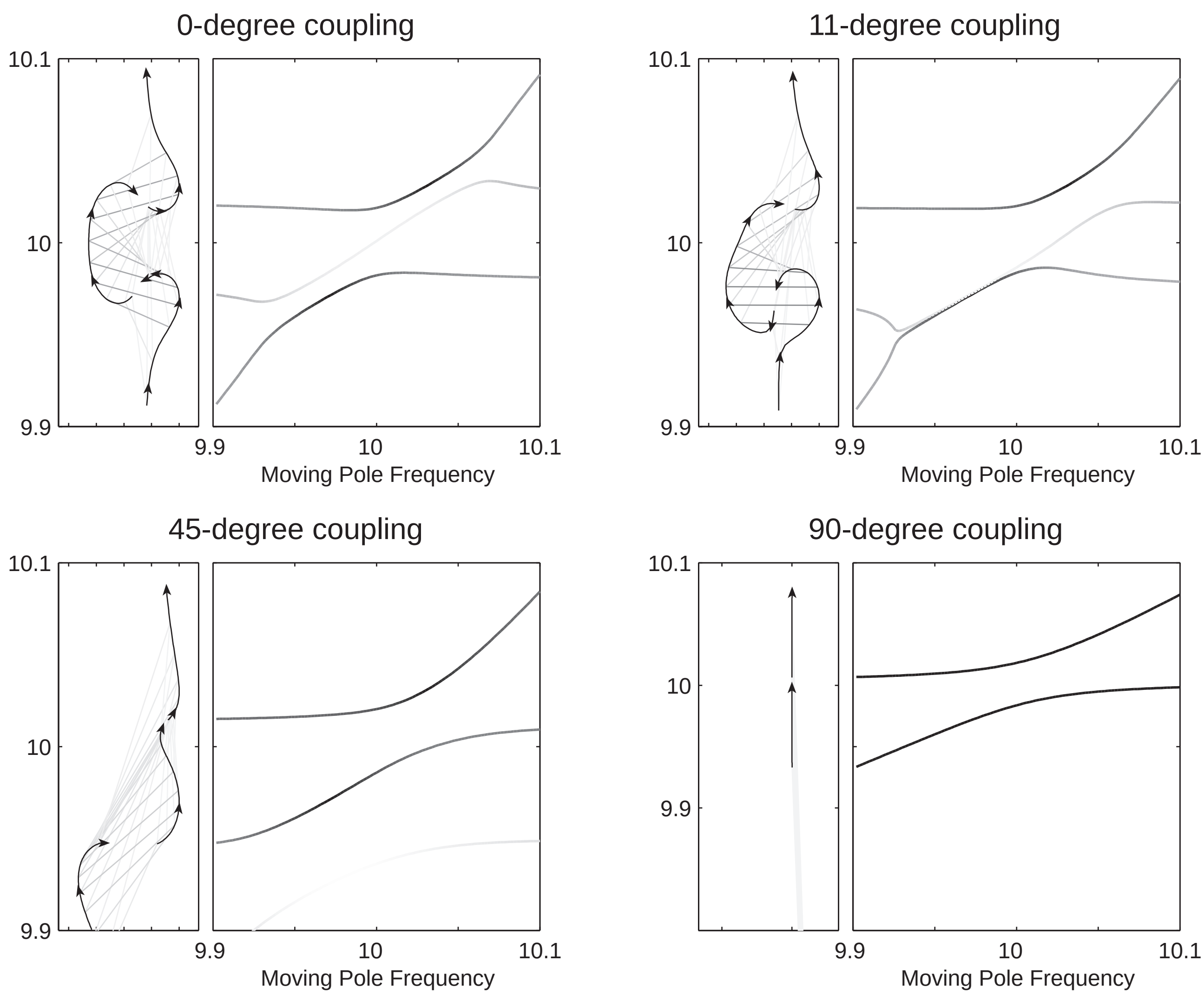
Three open-loop poles,
two fixed at ω=10 ± 0.03
one moving from ω=9.95 to 10.05
Weak Coupling



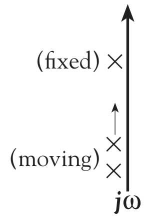
Here, the coupling is weak enough for the system to essentially act like two separate two-pole couplings as the mistuned pole moves through the fixed poles. If the fixed poles were closer, we would get a more interesting coupling, which would be the same effect as having a stronger coupling, which is the case on the next page.



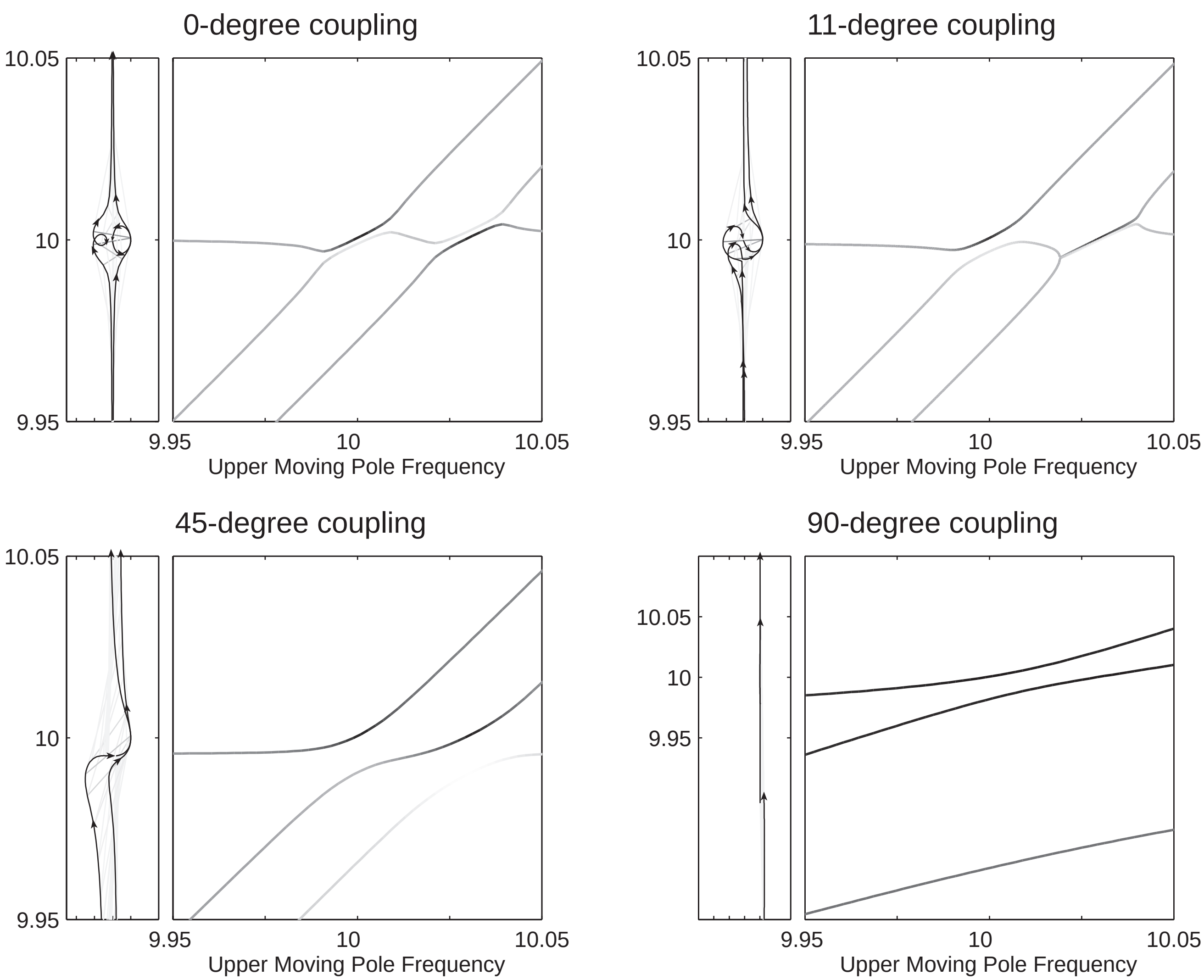
Three open-loop poles,
two fixed at ω=10 ± 0.03
one moving from ω=9.9 to 10.1
Strong Coupling



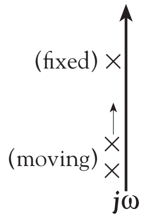
Here, the stronger coupling causes all three of the poles to affect each other, which gives a more complicated coupling behavior. An important thing to notice is that we have, in most cases, two poles at slow decay and the other at fast decay. This has two implications for the decay: there are still usually only two stages of decay: the first stage is determined by the fast decaying pole, and the two other poles, with similar decay rates, will constitute the second stage, but will beat against each other during the decay. This effect was noted in the papers by Nakamura and Hundley.



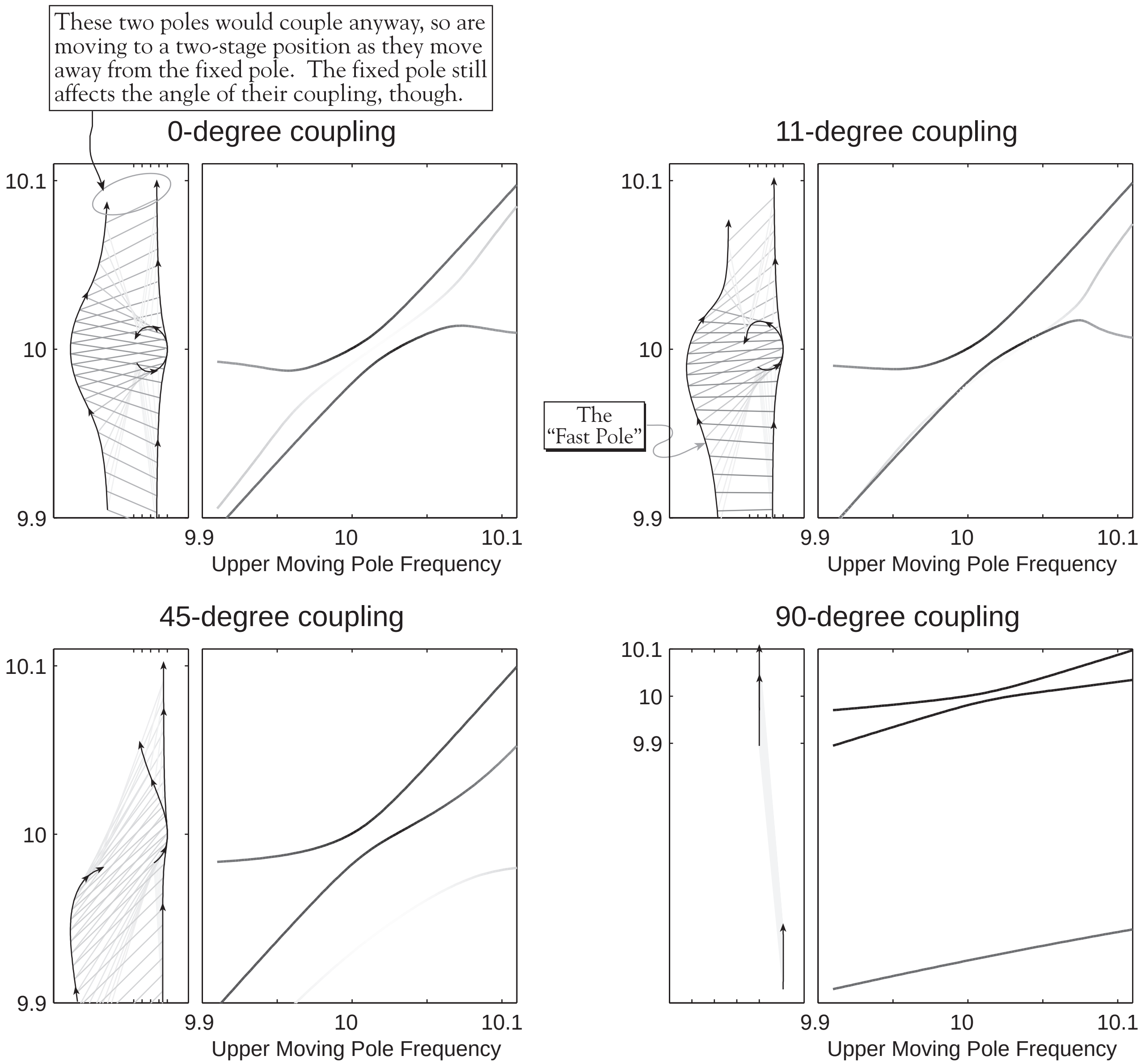
Three open-loop poles,
one fixed at ω=10
two moving from ω=9.95 & 9.92 to 10.05 & 10.02
Weak Coupling



Here we get essintially the same effects as in the previous weak-coupling figure (2 pages back), but the two-pole couplings occur in the same frequency ranges, making the *s*-plane figures more complicated. We can see in the Weinreich diagrams, though, that the couplings are still mostly unaffected by each other.



Three open-loop poles,
one fixed at ω=10
two moving from ω=9.9 & 9.87 to 10.1 & 10.07
Strong Coupling



If we compare this case to the previous strong-coupling case (two pages back), the major difference is that the poles are much closer to each other at their closest approach, which has the effect of making the coupling even stronger. Thus the fast-decay pole is further out to the left than before, and in this case the two moving poles have partially coupled together, so that even when they are far from the fixed pole, the fast pole stays at a quick decay.

Two-Strings, Real Coupling

Root Loci of a system as on P. 4, looking at the first 10 harmonic pairs. In this case, G_{load} is a real scalar. Note how at higher harmonics, the effects of coupling are “weaker”.

$delay_i = e^{-sT_i}$

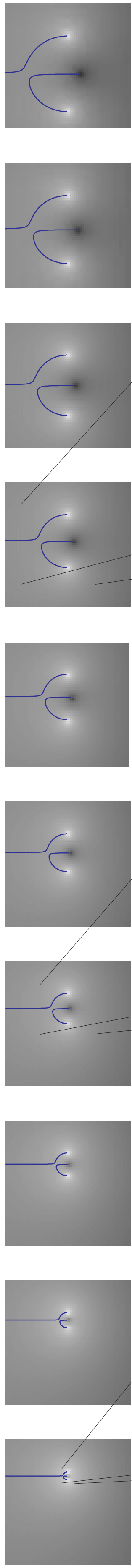
$G_{Forward} = \sum_i \frac{-e^{-sT_i}}{1 - e^{-sT_i}}$

$GH = G_{Forward} G_{load}$

$T_1 = 1.00$
$T_2 = 1.02$

Real Coupling:
$G_{load} = 1$

Note: All have the same shape (because the coupling is a real scalar), but different sizes. This is because the higher harmonics are further apart, thus the higher harmonics will couple less strongly.

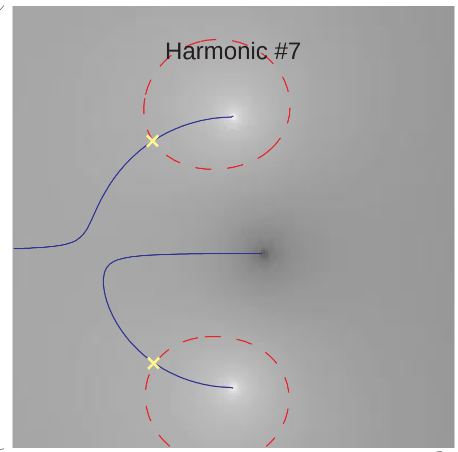


For a given value of coupling (k), the poles land at the intersections of the solid and dotted lines.

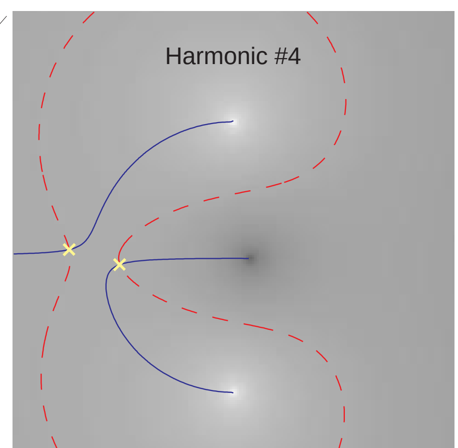
Dotted: $|GH| = {}^1/_k$ (k=0.2)

Solid: $\angle(GH) = 0$

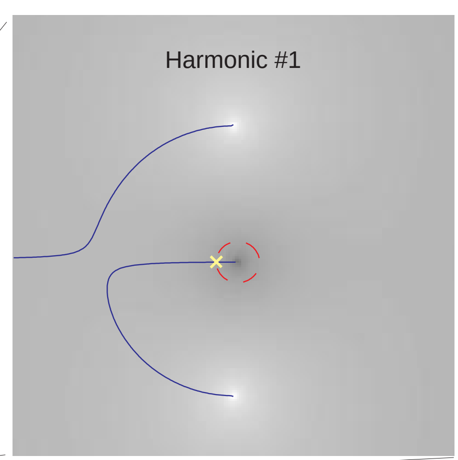
(remember, for string coupling, we use the 0° root-locus)



At high harmonics:
weak coupling
⇒ not in 2-stage decay.



At mid harmonics:
moderate coupling
⇒ barely 2-stage decay.

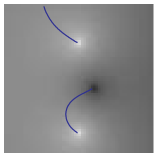


At low harmonics:
strongest coupling
⇒ well into 2-stage decay.

Two-Strings, 1-Pole Coupling

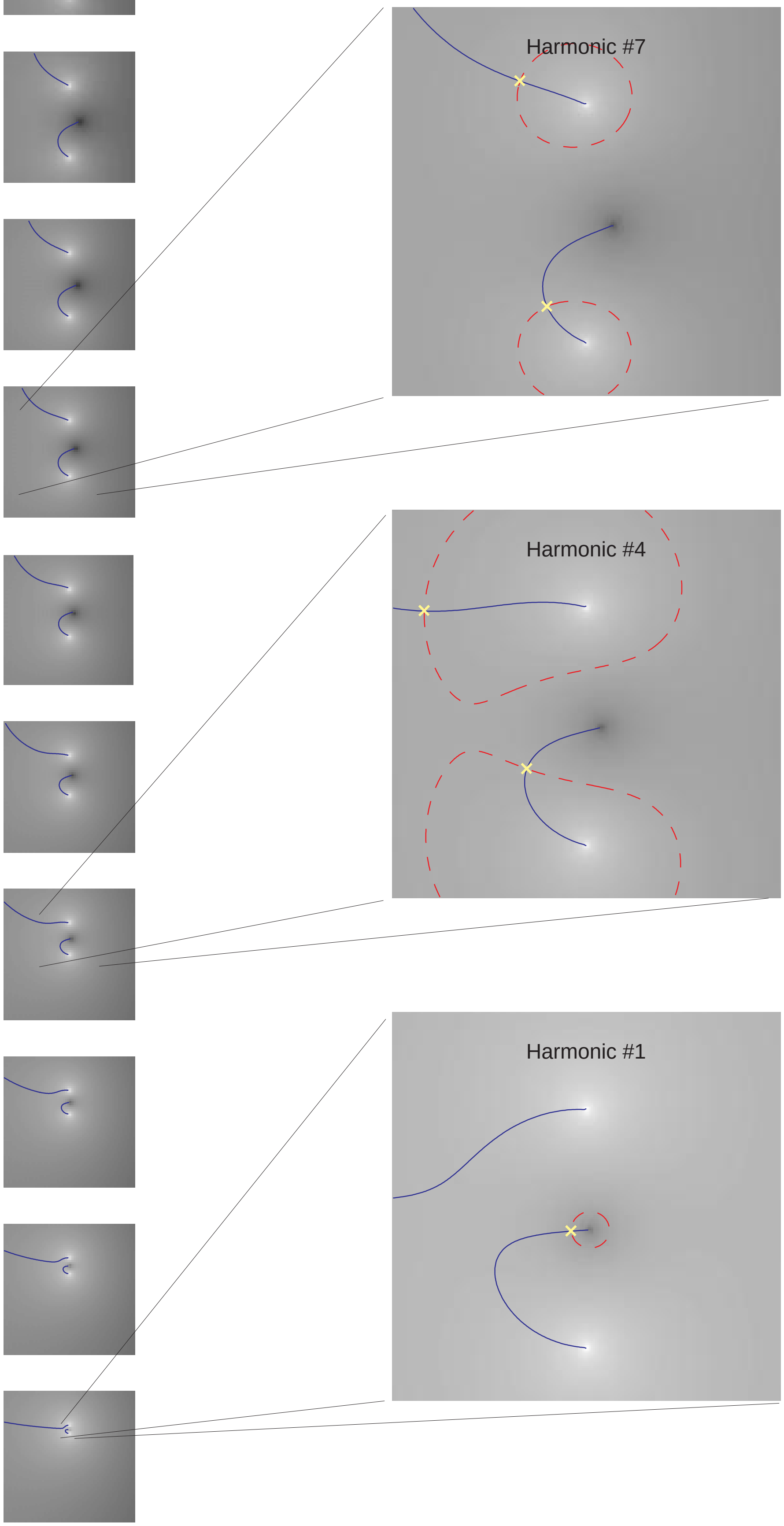
One-Pole Coupling:

$$G_{load} = \frac{100}{s+100}$$



In this case, G_{load} is a one-pole filter. This causes the coupling to be frequency dependant. i.e., each harmonic has a different coupling behavior.

At low frequencies, the coupling filter has a phase of essentially 0° (i.e. real). The coupling phase approaches 90° at high harmonics, with correspondingly different coupling behaviors.



For a given value of coupling (k), the poles land at the intersections of the solid and dotted lines.

Dotted: $|GH| = {}^{1}/_{k}$ (k=0.2)

Solid: $\angle(GH) = 0$

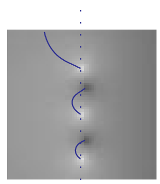
(remember, for string coupling, we use the 0° root-locus)

Three-Strings, 1-Pole Coupling

One-Pole Coupling:

$$G_{load} = \frac{100}{s+100}$$

In this case, G_{load} is a one-pole filter. This causes the coupling to be frequency dependant. i.e., each harmonic has a different coupling behavior.



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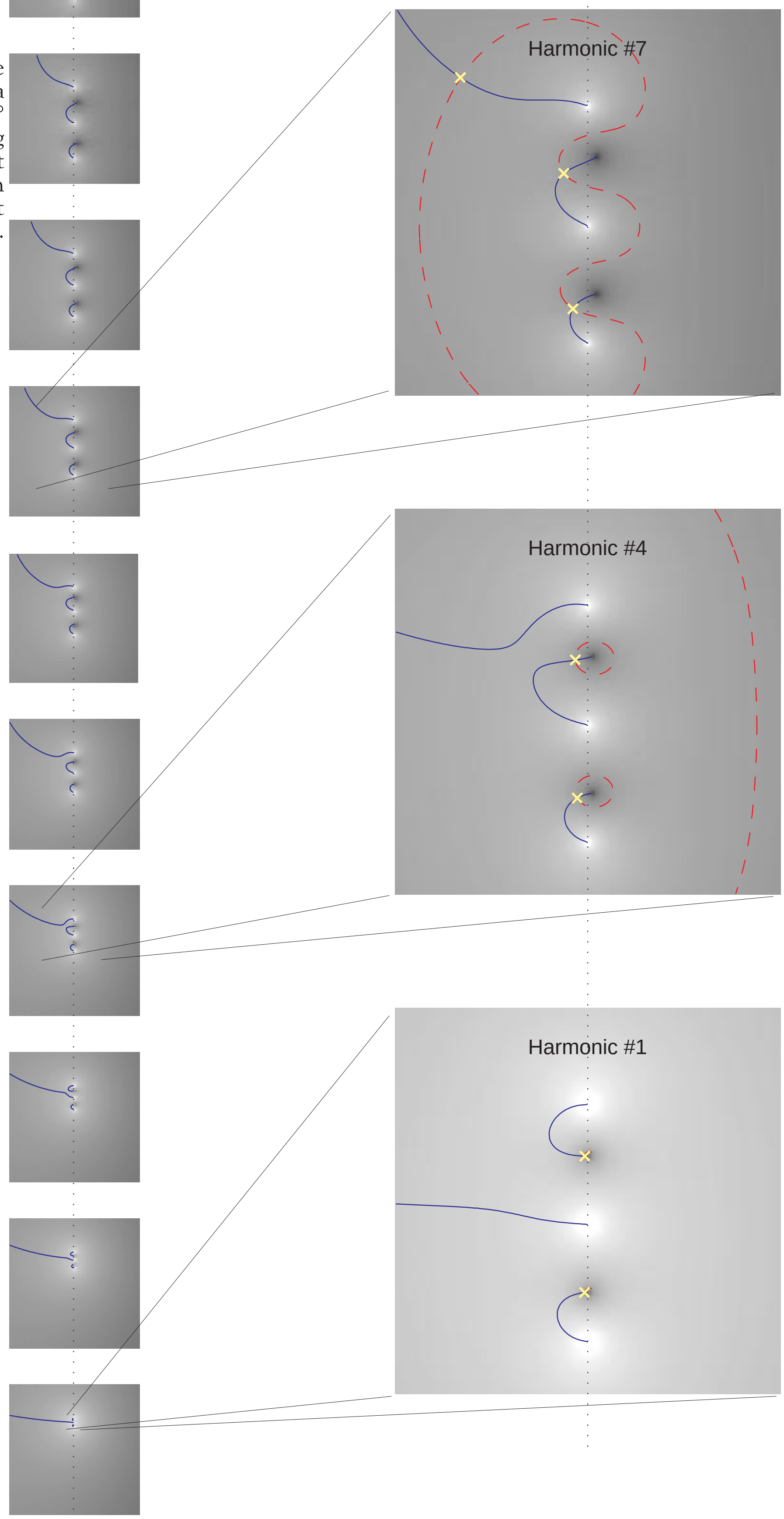


For a given value of coupling (k), the poles land at the intersections of the solid and dotted lines.

Dotted: $|GH| = {}^{1}/_{k}$ (k=0.2)

Solid: $\angle(GH) = 0$

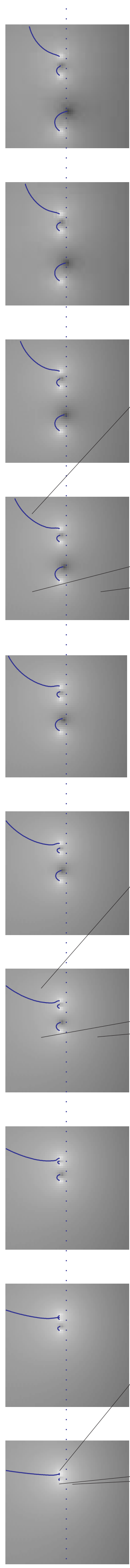
(remember, for string coupling, we use the 0° root-locus)

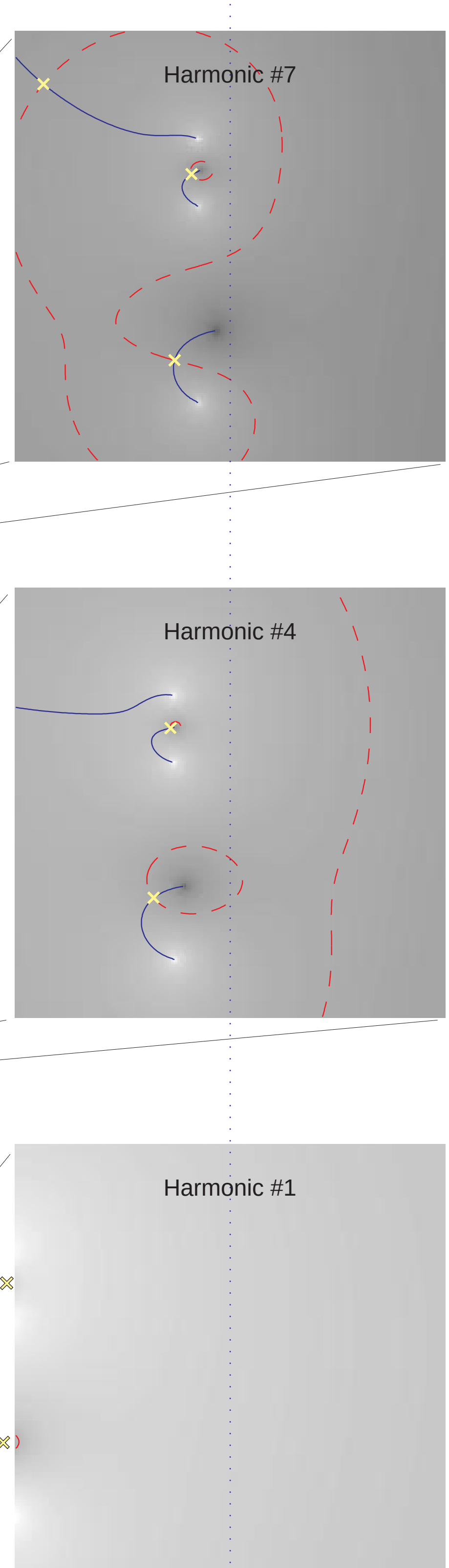
Three-Strings, 1-Pole Coupling, with Damping

In this case,The strings also have damping, which moves their poles off the jω axis.
This case also has a different detuning configuration than the previous figure.

One-Pole Coupling:

$$G_{load} = \frac{100}{s+100}$$





For a given value of coupling (k), the poles land at the intersections of the solid and dotted lines.

Dotted: $|GH| = {}^{1}/_{k}$ (k=0.2)

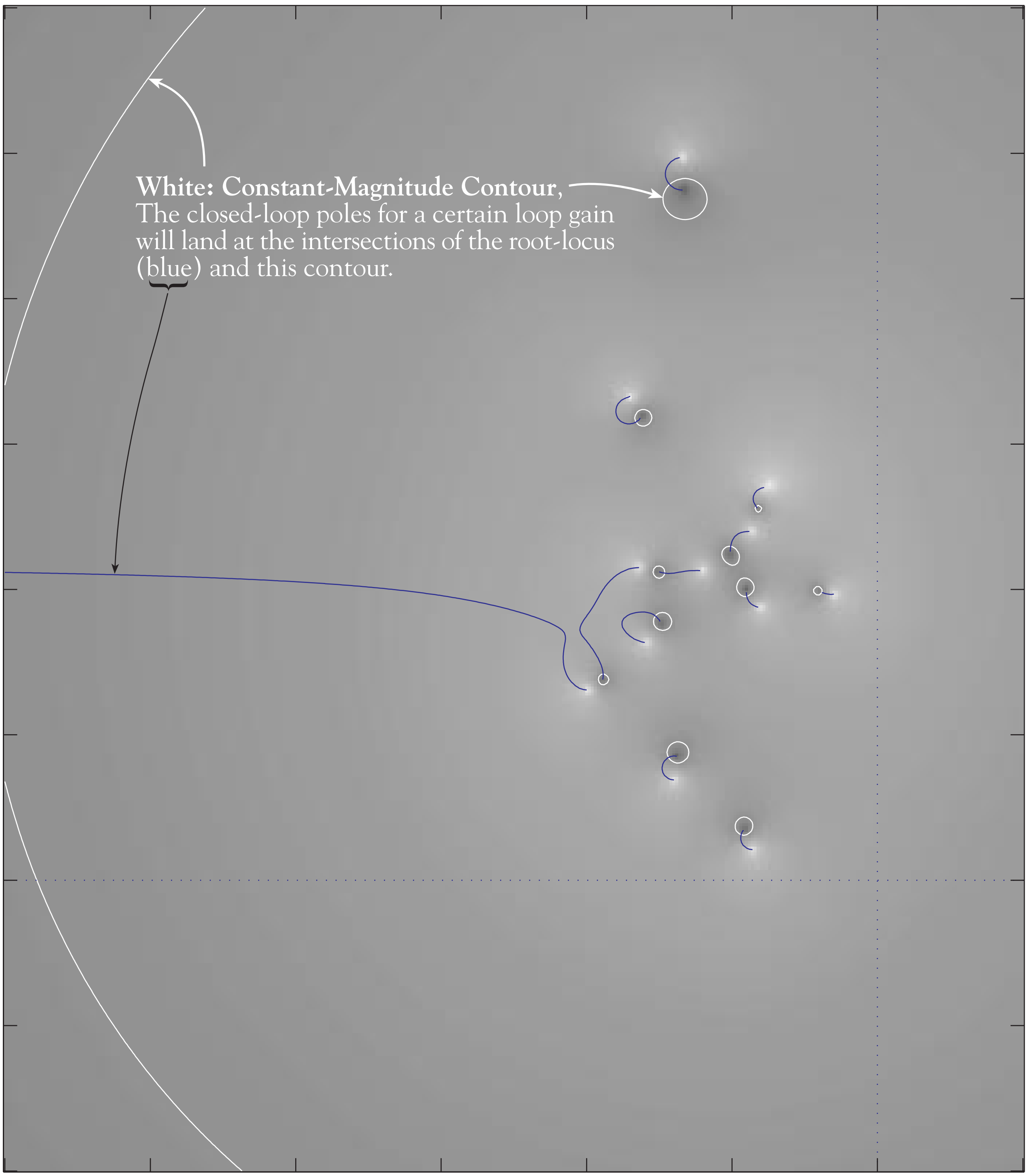
Solid: $\angle(GH) = 0$

(remember, for string coupling, we use the 0° root-locus)

Root-Locus of 13 summed random single-poles

$$G_{\text{Forward}} = \sum_i \frac{1}{s - p_i}$$

Note: There are 12 zeros, so all but one pole stay 'near' their open-loop positions ('slow decay'), and the other pole goes out to the left ('fast decay').



Conclusions

- **The Root Locus Provides Intuition on Coupling Behavior**
- **In 3-Mode Coupling, Two Modes Stay at Slow Decays**
  - **This Gives the Beating in the Second Stage**
- **Different String Harmonics Couple Differently**
  - **Higher Harmonics are More Detuned (Absolute Detuning, Not Relative).**
    This Makes the Higher Harmonics "Less Coupled".
  - **Frequency-Dependent Coupling Causes each Harmonic to Couple Differently**
    Each Harmonic Couples with a Different Coupling Phase Angle.

References

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[4] Franklin, G., Powell, J.D., Emami-Naeini, A. 1994. *Feedback Control of Dynamic Systems,* New York: Addison-Wesley, pp. 243-336.

[5] Smith, J.O. 1993. "Efficient Synthesis of Stringed Musical Instruments", ICMC93, Tokyo, pp.64-71.