Analysis Of Resonant Ring Modulation

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Here we analyze the harmonic content of $x(t)$ when resonant ring modulation is applied with the adaptive/PI-controller for integer n. We will assume that the instrument and effect parameters are not changing, and we will find candidate equilibrium states at time $t \rightarrow \infty$. We will continue to assume that the modes of the instrument line up perfectly in a harmonic series. Once the system is in equilibrium, we have the following:

$$x(t) = g(t) * (Lx(t)cos(2\pi n f_0 t)), \tag{1}$$

where L is the loop gain, $g(t)$ is the inverse Laplace transform of $G(s)$, and $*$ is the convolution symbol. Since n is an integer, $x(t)$ will be periodic with period $1/f_0$, and so we may describe $x(t)$ by its Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} X(k) e^{2\pi j k f_0}. \tag{2}$$

Let $g_k = G(2\pi j k f_0)$, and note that the g_k are real since the sensor and actuator are collocated. Now we may transform (1) into the frequency domain.

$$X(k) = \frac{Lg_k}{2}(X(k-n) + X(k+n)) \tag{3}$$

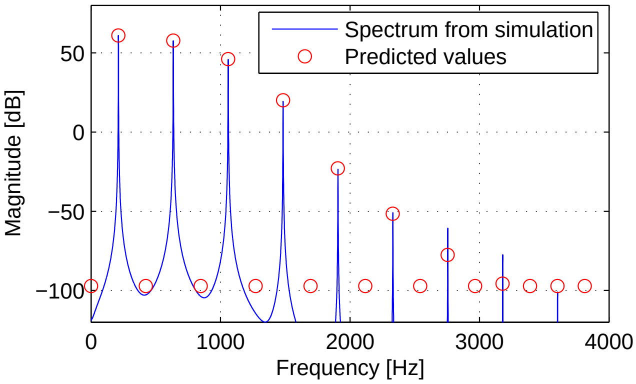
In practice, this relation may be approximated for $X(k)$ up to some Mth order since for real systems, $g_k \rightarrow 0$ as $k \rightarrow \infty$. The truncated expression may thus be expressed using linear algebra. Only the positive half of the spectrum is required since the signals as real, implying that $X(-k) = [X(k)]^*$, where $[]^*$ denotes the complex conjugate operation.

$$\bar{x} = (\; X(0) \;\; \Re X(1) \;\; \Im X(1) \;\; \Re X(2) \;\; \Im X(2) \;\; \ldots \;)^T \tag{4}$$

$$H_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots\vdots
\end{pmatrix} \tag{5}$$

$$\bar{x} = \frac{L}{2}\text{diag}(g_0 g_1 g_1 g_2 g_2 \ldots) H_n \bar{x} \tag{6}$$

If we let $Z_n = (\frac{1}{2}\text{diag}(g_0 g_1 g_1 g_2 g_2 \ldots) H_n)^{-1}$, then we may write the following:



1: Comparison of spectra from resonant ring modulation

$$Z_n \bar{x} = L \bar{x} \tag{7}$$

We see that $\bar{x}$ may be any eigenvector of Z_n, where the loop gain L is the corresponding eigenvalue. Eigenvectors that do not have a low-pass characteristic may be eliminated because they do not satisfy the low-pass assumption. Fig. 1 shows the match between the spectrum 60 seconds into a simulation using $n = 2$ and one of the remaining eigenvectors $\bar{x}$. The eigenvector finder does not have quite as large of a dynamic range as the simulation, and the spectrum from simulation shows some signs of mismatch at the 13th and 15th harmonics. This is due to harmonic distortion from L varying with time and perhaps also due to the Mth-order approximation of the spectra; however, this mismatch is probably inaudible.